# Problemen

| Problem Section

This Problem Section is open to everyone. For each problem the most elegant correct solution will be rewarded with a 20 Euro book token. The judges reserve the right to withdraw the prize if none of the solutions is deemed worthy. The problems and results can also be found on the Problem Section website www.nieuwarchief.nl/ps.
Occasionally there will be a Star Problem, of which the editors do not know any solution. Whoever first sends in a correct solution within one year will receive a prize of 100 Euro.
Both suggestions for problems and solutions can be sent to uwc@nieuwarchief.nl or to the address given below in the left-hand corner; submission by email (in LaTeX) is preferred. When proposing a problem, please include a complete solution, relevant references, etc. Group contributions are welcome. Participants should repeat their name, address, university and year of study if applicable at the beginning of each problem/solution. If you discover a problem has already been solved in the literature, please let us know. The submission deadline for this edition is September 1, 2008.

**Problem A** (proposed by Ovidiu Furdui)
Denote the fractional part of a positive real number $x$ by $\{x\}$, for example $\{\pi\} = \pi - 3$. Evaluate the following double integral:

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} dx dy.$$

**Problem B** (Folklore)
Let $S$ be a set consisting of 15 integers, and such that for all $s \in S$ there exist $a, b \in S$ with $s = a + b$.

1. Show that there exists a non-empty subset $T \subset S$ of at most seven elements that add up to 0.
2. Show that this does not need to be true for $S$ with 16 elements.

**Problem C** (Proposed by the Arithmetic Geometry group of Leiden University)
Let $f : \mathbf{R} \to \mathbf{R}$ be a $C^\infty$ function (that is, all higher derivatives of $f$ exist and are continuous) such that

1. $f(x) = 0$ if $x \leq 0$,
2. $f(x) > 0$ if $x > 0$.

Is it true that $\sqrt{f} : \mathbf{R} \to \mathbf{R}_{\geq 0}$ is a $C^1$ function (that is, that its derivative exists and is continuous)?

**Edition 2007/3**
For Edition 2007/3 we received submissions from Birgit van Dalen (Leiden), Reza Takapoui (Delft), Kee-Wai Lau (Hong Kong), Michiel Vermeulen *et al.* (Zwaanshoek), Konstantine Zelator (Toledo, Ohio).

**Problem 2007/3-A** Let $a$ be an integer. Let $(x_n)_n$ be the sequence determined by $x_1 = a$ and $x_{n+1} = 2x_n^2 - 1$. Show that $n$ and $x_n$ are coprime for all $n$.

**Solution** We present the solution given (independently) by Birgit van Dalen and Kee-Wai Lau. This problem was also solved by Konstantine Zelator.
We must show that for all primes $p$ and for all $k > 0$ we have that $p$ does not divide $x_{kp}$. Consider the sequence modulo a prime $p$. If 0 mod $p$ does not occur in the sequence then we are done. If not, there is a smallest positive integer $m$ such that $x_m \equiv 0$ modulo $p$. It follows that $x_i$ is not congruent to 1 or $-1$ modulo $p$ for $1 \leq i < m$ for otherwise we would have $x_j \equiv 1$ modulo $p$ for all $j > i$. Also, the $x_1, \ldots, x_{m-1}$ are pairwise distinct mod $p$, since if $x_i \equiv x_j$ with $1 \leq i < j < m$ then from $i$ on the sequence mod $p$ is periodic with period $j - i$, contradicting the minimality of $m$. We conclude that $m = 1$ if $p = 2$

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and $m \leq p-2$ if $p$ is odd. It now suffices to observe that $x_{m+1} \equiv -1$ and $x_{m+k} \equiv 1$ for all $k > 1$.

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**Problem 2007/3-B** Let $G$ be a group with $n$ elements and $S \subset G$ a non-empty subset. Show that the set

$$S^n = \{s_1 s_2 \cdots s_n | s_i \in S\}$$

is a subgroup of $G$.

**Solution** This problem was solved by Michiel Vermeulen *et al*. and Reza Takapoui. Their solutions are similar to one another and to the one we present here.
Choose an element $s$ of $S$. Since $sS^i \subset S^{i+1}$ we have that the sequence of cardinalities $\#S$, $\#S^2$, $\#S^3$, ... is non-decreasing. Note that if $\#S^i = \#S^{i+1}$ then

$$\#S^{i+1} = \#S^{i+2} = \#S^{i+3} = \cdots.$$

Indeed, this is clear from

$$sS^{i+1} = sS^i S = S^{i+1} S = S^{i+2}.$$

As the cardinalities are bounded by $n$, it follows that

$$\#S^n = \#S^{n+1} = \cdots.$$

Clearly $e = s^n$ lies in $S^n$, hence $S^n \subset S^{2n}$. By the above we have $S^n = S^{2n}$ and hence it follows that $S$ is closed under multiplication and that $S$ is a subgroup.

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**Problem 2007/3-C** **(a)** Given 2007 points in the plane such that no pair has distance strictly less than one, show that one can find a subset of 288 points in which no pair has distance strictly less than $\sqrt{3}$.
**(b)*** Supposedly the number 288 in part (a) is not optimal. Find upper and lower bounds for the optimal value.

**Solution** Unfortunately we received no submissions for Problem C. We present our own solution to (a). We would like to stress that the Star Problem (b) is still open for competition, with a 100 euro book token prize. Deadline: September 1, 2008.
First we introduce some notation. Given a finite set $S \subset \mathbf{R}^2$ such that no pair of elements of $S$ has distance strictly less than one we define $f(S)$ as the maximum size of a subset in which no pair of elements has distance strictly less than $\sqrt{3}$. For a positive integer $n$ we define $f(n)$ to be the minimum of the $f(S)$ over all $S$ of cardinality $n$.
**Theorem.** *For $n \geq 6$, the inequalities*

$$\left\lfloor \frac{n+9}{7} \right\rfloor \leq f(n) \leq \lceil n/5 \rceil$$

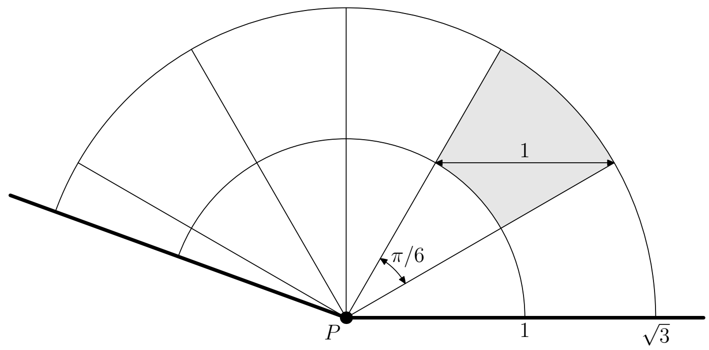
*hold.*
In particular this gives $288 \leq f(2007) \leq 402$.
To see the upper bound: take $S$ to consist of clusters of 5 points (plus a residue cluster if necessary). The main ingredient in he proof of the lower bound is the following lemma.

**Lemma.** $f(n+7) \geq f(n) + 1$ *for all* $n > 0$.
*Proof of the lemma*
Let $S$ be such that $\#S$ equals $n+7$ and $f(S)$ is minimal, that is $f(S) = f(n+7)$. Pick a point $P \in S$ on the boundary of the convex hull of $S$. Denote by $S' \subset S$ the subset of

points that have distance at least $\sqrt{3}$ from $P$.

*Claim:* $\#S' \geq \#S - 7 = n$.

Indeed, see the picture: in any area such as the shaded one there can be at most one point of $S - S'$.




Now to finish the proof of the lemma consider a $T' \subset S'$ of cardinality $f(S')$ and such that the distance between any pair of distinct points of $T'$ is at least $\sqrt{3}$. Then the same is true for $T := T' \cup \{P\} \subset S$ and hence

$$f(n+7) = f(S) \geq \#T = f(S') + 1 \geq f(n) + 1$$

where the last inequality follows from the claim and from the fact that $f$ is a non-decreasing function of $n$.

*Proof of the lower bound*

By the lemma it suffices to show $f(6) \geq 2$ and $f(12) \geq 3$. This follows from essentially the same argument as the one given in the lemma, using that given a set $S$ of 12 (resp. 6) points in the plane there is always a point $P$ on the boundary of its convex hull such that the boundary forms an angle of at most $5\pi/6$ (resp. $2\pi/3$) at $P$.