Hans Duistermaat
Mathematisch Instituut Universiteit Utrecht
Postbus 80 010, 3508 TA Utrecht
duis@math.uu.nl

Abel prize winner 2006: Lennart Carleson

Achievements until now

What conjecture will Lennart Carleson attack next? Lennart A.E. Carleson, 2006 Abel Prize winner is known for at least three cornerstone achievements in mathematical analysis. The proofs of these longstanding conjectures are all very complicated. At the Dutch Mathematical Conference 2007 in Leiden, Hans Duistermaat, professor in mathematics at the Utrecht University and one of the great specialists in the field of Fourier theory, explains Carleson's results.

Lennart Axel Edvard Carleson was born in 1928. He obtained his PhD in 1950 in Uppsala with Arne Beurling with the thesis, *On a class of meromorphic functions and its exceptional sets*. He has been professor at the University of Uppsala (1955–1993) and the University of California at Los Angeles, and has had 26 PhD students. He was director of the Mittag-Leffler Institute in Djursholm from 1968–1984, where he placed special emphasis on stimulating young mathematicians, and was president of the International Mathematical Union 1978–1982.

He has been awarded the Leroy Steel Prize (AMS) in 1984, the Wolf Prize in Mathematics in 1992, the Lomonosov Gold Medal in 2002, the Sylvester Medal in 2003, and the Abel Prize in 2006. The latter "for his profound and seminal contributions to harmonic analysis and the theory of smooth dynamical systems."

MathSciNet has 68 matches. His most famous ones are:

- 'Interpolations by bounded analytic functions and the *corona problem*', *Annals of Math* **76** (1962), pp. 547–559.
- 'On the convergence and growth of partial sums of *Fourier series*', *Acta Math* **116** (1966), pp. 135-157.
- (with M. Benedicks) 'The dynamics of the *Hénon map*', *Annals of Math* **133** (1991), pp. 73–169.

The corona theorem

According to the uniformization theorem, every simply connected complex one-dimensional complex analytic manifold which admits a non-constant *bounded* holomorphic function is conformal to the open unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane. The set B of all bounded holomorphic functions on D forms a Banach algebra, when provided with the supremum norm.

Let $f_1, \ldots, f_n \in B$ and let I denote the ideal in B generated by $f_1, \ldots, f_n$, so I is the set of all $\sum_{i=1}^n g_i f_i$ where $g_i \in B$. We have $I = B$ if and only if there exist $g_i \in B$ such that $\sum_{i=1}^n g_i f_i = 1$. If there exists $z \in D$ such that $f_i(z) = 0$ for $1 \leq i \leq n$, then $h(z) = 0$ for every $h \in I$, and I is a proper ideal in B.

If the f_i do not have a common zero in D, but if there exists a sequence $z_j \in D$ such that $f_i(z_j) \to 0$ for $j \to \infty$, for every $1 \leq i \leq n$, then $\forall h \in I\ h(z_j) \to 0$ for $j \to \infty$, and again I is a proper ideal in B. Note that in this case $|z_j| \to 1$, and by passing to a subsequence we can arrange that the z_j converge to a point on the boundary $\partial D = \{z \in \mathbf{C} : |z| = 1\}$ of D. The case which remains is that there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f_i(z)| \geq \delta \tag{*}$$

for every $z \in D$, when h belonging to I does not correspond to h having a zero in D or, in the above sense, in ∂D, so $h \in I$ would be a 'corona property' when the disk of the sun is eclipsed by the moon.

Carleson proved that condition (*) implies that $I = B$, which settles the matter.

Fourier series

Limits of sums

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx},$$

where the coefficients $c_k \in \mathbf{C}$, $k \in \mathbf{Z}$, are given, and $n \to \infty$ appeared in the 18-th century. In the beginning of the 19-th century Fourier very convincingly demonstrated their usefulness in analysis and, among others, proved that if $s_n(x) \to f(x)$ as $n \to \infty$ in a weak sense, then

$$c_k = \frac{1}{2\pi} \int_{\mathbf{R}/2\pi\mathbf{Z}} f(x) e^{-ikx} dx, \qquad k \in \mathbf{Z}.$$

The right hand side is called the k-th Fourier coefficient $c_k(f)$ of an arbitrary integrable function f (or even 2π-periodic distribution f). In this way each integrable 2π-periodic function has its Fourier series, and around 1800 one had the camp of people who believed that 'every' function f was equal to its own Fourier series, and the skeptics who couldn't believe this.

In 1829 Dirichlet gave a convergence proof which worked for every continuous function f which is piecewise monotonic. He suggested that every continuous function is piecewise monotomic, which is a bit strange, because it is easy to construct a convergent Fourier series which is not monotomic on any subinterval.

Then in 1876 Du Bois-Reymond constructed a continuous 2π-periodic function f such that for every x in a countable dense subset E of $\mathbf{R}/2\pi\mathbf{Z}$ the partial sums $s_n(x)$ are not even bounded. This shows that, for a given integrable function f on $\mathbf{R}/2\pi\mathbf{Z}$, the problem for which $x \in \mathbf{R}/2\pi\mathbf{Z}$ the $s_n(x)$ converge to $f(x)$ is quite subtle. Carleson's proof that, when $f \in L^2(\mathbf{R}/2\pi\mathbf{Z})$, the set of $x \in \mathbf{R}/2\pi\mathbf{Z}$ such that the $s_n(x)$ do not converge to $f(x)$ has *zero Lebesgue measure*, was a breakthrough in harmonic analysis.

The Hénon map

The Hénon map is defined by the transformation

$$T\ :(x,y) \mapsto (1+y-ax^2, bx)$$

from the plane to itself, with $a, b \in \mathbf{R}$ as parameters. It had been introduced in 1978 by Hénon, with $a = 1.4$ and $b = 0.3$, as a simple example where computer simulations indicate the existence of a strange attractor for iterations of T, similar to those obtained by Lorenz in 1963 in his higher dimensional weather models. The simulations with such models indicate not only parameter values for which the dynamical system has chaotic behaviour, but also open regions in the parameter space with finite periodic orbits as attractors, and these open regions even could be dense in the parameter space. Sheldon and Newhouse proved that (long-)periodic attractors are topologically (not in measure) generic, and actually it turned out to be very hard to *prove* that the set of parameter values, for which the iterates of the transformation show chaotic behaviour, has positive Lebesgue measure.

Benedicks and Carleson studied the dynamics of the Hénon map for small positive values of b and for a close to 2. In this case the Hénon map is a small perturbation of the one-dimensional quadratic map

$$x \mapsto 1 - ax^2 \qquad (**)$$

for which Jakobson in 1981 had proved chaotic behavior for a set of parameter values a with positive Lebesgue measure. Using Lyapunov exponential estimates for (**) with a close to 2, Benedicks and Carleson proved the following theorem.

Theorem. *Let W^u be the unstable manifold of T at its fixed point in $x > 0$, $y > 0$. Then for every $c < \log 2$ there exists a number $b_0 > 0$ such that for all $0 < b < b_0$ there exists a set $E(b) \subset \mathbf{R}$ of positive Lebesgue measure such that for all $a \in E(b)$ the following statements hold.*

i. There is a nonempty open U such that for all $z \in U$: $dist(T^v(z), \overline{W^u}) \to 0$ as $v \to \infty$. Moreover, the domain of attraction of $\overline{W_u}$ has non empty interior.

ii. There exists an element $z_0 \in W^u$ such that $\{T^v(z_0)\}_{v=0}^{\infty}$ is dense in W^u, and
$$||DT^v(z_0)(0,\ 1)^t|| \geq e^{cv}.$$

The above theorem implies strange attractors for each pair of numbers a and b satisfying $0 < b < b_0$, $a \in E(b)$.

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Lennart Carleson receives the Abel prize from Queen Sonja of Norway

Later developments

For each of the three problems, the subject had already intensively been studied when Carleson entered, and the statements had been conjectured, with the recognition that it looked very difficult to prove these. Therefore one cannot say that Carleson invented the subject or the theorems, but that in each case his contribution consisted of the introduction of techniques which were powerful enough to prove these basic theorems in the subject. Because these techniques are the main contributions of Carleson, I would have loved to be able to explain these to you in sufficient detail here, so that you would really understand what is going on. However, I have to admit that trying to understand Carleson's proofs, I soon realized that it would probably take months of concentrated work for me to do so, and because I did not have that time, I had to admit defeat. It was only a slight consolation that in later articles on the subject many of the specialists in the subject also found Carleson's proofs to be technically very difficult, but at the same time his techniques to be very powerful, notably for the proof of the one particular theorem. Therefore other proofs were also of great interest to the specialists. I mention, regarding later developments, the following articles.

- T.W. Gamelin, 'Wolff's proof of the corona theorem', *Israel J. Math.* **37** (1980), pp. 113-119, which unlike Carleson's proof, is based on the use, suggested by Hörmander in 1967, of estimates for the Cauchy-Riemann operator $\partial/\partial\bar{z}$. E. L. Stout: "This proof is a gem of classical analysis".
- R. A. Hunt, 'On the convergence of Fourier series', pp. 235-255 in the *Proc. Conf. on Orthogonal Expansions and their Contiuous Analogues, Edwardsville, 1967*, Southern Illinois Univ. Press, Carbondale (1968) shows that almost everywhere convergence of Fourier series is true for any $f \in L^P$, $p > 1$.
- C. Fefferman, 'Pointwise convergence of Fourier series', *Annals of Math.* **98** (1973) pp. 551-571 contains a new proof of Hunt's theorem which has been very influential.
- C. Thiele, 'Wave Packet Analysis', *CBMS Reg. Conf. Series* **105**, AMS, Providence (2006), Chapter 7, contains a very nice exposition in terms of wave packets.

There are quite many more recent papers in dynamical systems which build on the methods of Benedicks and Carleson, for example [3]. However, I have not seen an essentially new proof the theorem of Benedicks and Carleson, where also by the specialists B and C's proof is characterized as 'a true tour de force'. Instead I found an article by Dobrynskii in the Doklady 2004 [2], in which it is claimed that the Benedicks-Carleson set is empty, in flat contradiction with the B-C theorem. The Math. Reviews reviewer M.L. Blank drily remarks that "unfortunately D's paper is written in a way that makes it very difficult, if not impossible, to check the claim". As the Abel prize was awarded in 2006 to Carleson for, among others, "his profound and seminal contributions to the theory of smooth dynamical systems", I surmise that the Abel prize committee had sufficient support from the specialists to be convinced that the Benedicks-Carleson theorem is correct.

Carleson's proof of the corona theorem

I would like at least to convey *some* of the techniques in which Carleson excels. For the proof of the corona theorem, let $f_1, \ldots, f_n \in B$ and $\sum_{i=1}^{n} |f_i(z)|| \geq \delta > 0 \; \forall z \in D$. His proof is by induction on n. First assume that f_n has only finitely many simple zeros a_v in D and is bounded away from zero near ∂D. For z in a sufficiently small open neighborhood E of the zero set of f_n we have $|f_n(z)| \leq \frac{1}{2}\delta$, hence $\sum_{i=1}^{n-1} |f_i(z)| \geq \frac{1}{2}\delta$. Therefore if E is the union of disjoint simply connected domains, then in each of these the corona theorem holds for $f_1, \ldots, f_{n-1}$, which leads to bounded holomorphic functions $\gamma_1, \ldots, \gamma_{n-1}$ on E such that $\sum_{i=1}^{n-1} \gamma_i f_i = 1$ on E. Let $g_1, \ldots, g_{n-1}$ be bounded holomorphic functions on D such that $g_i(a_v) = \gamma_i(a_v)$ for all v. We can take the g_i as polynomials, but later in the proof it becomes essential to take $g_i \in B$ with minimal supremum norm. Now $g_n = (1 - \sum_{i=1}^{n-1} g_i f_i)/f_n$ is analytic in D because $1 - \sum_{i=1}^{n-1} g_i(a_v) f_i(a_v) = 1 - \sum_{i=1}^{n-1} \gamma_i(a_v) f(a_v) = 0$ for every v. The function g_n is bounded because f_n was assumed to be bounded away from zero. For any finite set of distinct point a_v in D the *Blaschke product*

$$A(z) = \prod_{v=1}^{s} \frac{a_v - z}{1 - z\overline{a_v}} \frac{\overline{a_v}}{|a_v|}$$

is the prototype of a bounded analytic function on D with simple zeros in the a_v. Actually $|A(z)| = 1$ when $|z| = 1$ and therefore $|A(z)| \leq 1$ when $|z| < 1$. For general $f_n \in B$, using suitable approximations of suitable B-multiples of f_n by means of Blaschke products, it is sufficient, in the above proof with f_n replaced by A, that the above interpolation problem $g(a_v) = \gamma(a_v)$, has solutions $g \in B$ with $||g|| < \delta^{-C}$, when the γ is holomorphic and $|\gamma| < 1$ in a neighborhood of the a_v where $|A(z)|$ is of order δ.

In a previous paper [4]Carleson had showed that the minimal norm for G is equal to

$$\sup \left\{ \left| \sum_{v=1}^{s} \frac{G(a_v)\gamma(a_v)}{A'(a_v)} \right|, G \text{ analytic on } D, ||G||_1 = 1, \right\}$$

where

$$||G||_1 = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})| d\theta.$$

On the other hand, Cauchy's integral formula yields

$$\sum_{v=1}^{s} \frac{G(a_v)\gamma(a_v)}{A'(a_v)} = \frac{1}{2\pi i} \int_\Gamma \frac{G(z)\gamma(z)}{A(z)} dz,$$

where Γ is a one-dimensional cycle which runs once around each zero a_v of A. We therefore have the desired estimate if we can arrange that

$$\delta^N < |A(z)| < \delta \text{ on } \Gamma, \tag{$\dagger$}$$

with $N > 1$ suitably chosen, and there is a constant M such that

$$\int_\Gamma |G(z)||dz| \leq M, \tag{$\dagger\dagger$}$$

for all G analytic on D such that $||G||_1 = 1$. Because the level curves of $|A(z)|$ can become too long for the arbitrary Blaschke products, these level curves in general cannot be taken as Γ. The heart of Carleson's proof is a very ingenious construction of Γ such that (†) and (††) holds. This shows Carleson's mastery in making *geometric* constructions which satisfy the needed analytic *estimates*, in situations where *no simple constructions* yield the desired estimate.

References

1 D.K. Arrowsmith, and C.M Place, *Dynamical systems*, Chapman & Hall, 330 pp., 1992.

2 V.A. Dobrynskii, 'On the existence of Hénon attractors' ,*Doklady Math.* **70** (2004), pp. 574-579.

3 M. Benedicks, 'Non uniformly hyperbolic dynamics: Hénon maps and related dynamical systems', *Int. Congr. of Mathematicians, Beijing 2002*, vol. III, pp. 255-264.

4 L. Carleson, 'An interpolation problem for bounded analytic functions', *Amer. J. Math.* **80** (1958), pp. 921–930.