# Problemen

| Problem Section

The *Universitaire Wiskunde Competitie* (UWC) has changed back into a general Problem Section, open for everyone. In particular, the prizes are no longer restricted to students. For each problem the most elegant solution will be rewarded with a 20 Euro book token. The problems and results can also be found on the Problem Section website www.nieuwarchief.nl/ps.
Now and then there will be a Star Problem, of which the editors do not know any solution. Whoever first sends in a correct solution within one year will receive a prize of 100 Euro. To round off Session 2006/2 correctly, this issue includes the ladder.
Both proposed problems and solutions can be sent to uwc@nieuwarchief.nl or to the address given below in the left-hand corner; submission by email (in LaTeX) is preferred. When proposing a problem, please include a complete solution, relevant references, etc. Group contributions are welcome. Participants should repeat their name, address, university and year of study if applicable at the beginning of each problem/solution. If you discover a problem has already been solved in the literature, please let us know. The submission deadline for this session is March 1, 2007.
The prizes for the Problem Section are sponsored by *Optiver Derivatives Trading*.



**Problem A** (Folklore)
Seventeen students play in a tournament featuring three sports: badminton, squash, and tennis. Any two students play against each other in exactly one of the three sports. Show that there is a group of at least three students who compete amongst themselves in one and the same sport.

**Problem B** (Proposed by Arthur Engel)
The sequence $\{a_n\}_{n\geq 1}$ is defined by

$$a_1 = 1; a_2 = 12; a_3 = 20; a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n \ (n \in \mathbf{N}).$$

Prove that $4a_na_{n+1} + 1$ is a square for all $n \in \mathbf{N}$.

**Problem C** (Proposed by Michiel Vermeulen)
Let $G$ be a finite group of order $p+1$ with $p$ a prime. Show that $p$ divides the order of $Aut(G)$ if and only if $p$ is a Mersenne prime, that is, of the form $2^n - 1$, and $G$ is isomorphic to $(\mathbf{Z}/2)^n$.

**Edition 2006/2**
For Session 2006/2 We received submissions from Toon Brans, Ruud Jeurissen, Jaap Spies, and Peter Vandendriessche.

**Problem 2006/2-A** Prove or disprove the following:
In a $9 \times 9$ Sudoku–square one randomly places the numbers $1 \ldots 8$. There is at least one field such that if any one of the numbers $1 \ldots 9$ is placed there, the Sudoku–square can be filled in to a (not necessarily unique) complete solution.

**Solution** This problem was solved by Toon Brans, Ruud Jeurissen, and Peter Vandendriessche. Their solutions were comparable.
We disprove the statement as follows: If the numbers $1 \ldots 8$ are placed on the main diagonal from the top left corner towards the bottom right, there is no field where a number between 1 and 8 can be placed without giving a contradiction. In the first row the 1 cannot be placed, in the second row no 2 is possible, ..., in the eighth row no 8. This is the case for the columns as well. In the field in the lower right corner no 8 is possible.

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# Oplossingen | Solutions

**Problem 2006/2-B** Imagine a flea circus consisting of $n$ boxes in a row, numbered 1, 2,. . ., $n$. In each of the first $m$ boxes there is one flea ($m \leq n$). Each flea can jump forward to boxes at a distance of at most $d = n - m$. For all fleas all d+1 jumps have the same probability.
The director of the circus has marked $m$ boxes as special targets. On his sign all $m$ fleas jump simultaneously (no collisions).
1. Calculate the probability that after the jump exactly $m$ boxes are occupied.
2. Calculate the probability that all $m$ marked boxes are occupied.

**Solution** No solutions were sent in. The solution below is based on that of the proposer Jaap Spies.

*Part 1*
The jumps of the fleas correspond to a bipartite graph $G$. The possible jumps of the fleas can be coded in a (0,1)-matrix $B$ of size $m$ by $n$, with $b_{ij} = 1$ if and only if $i \leq j \leq i + d$. The total number of jumps with exactly $m$ boxes occupied is the same as the number of injective maps $\pi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ for which $b_{i\pi(i)} = 1$ for all $i$. This number equals per $(B)$, the permanent of $B$, see [1], p. 44. The probability that after the jump exactly $m$ boxes are occupied is therefore per $(B)/(d+1)^m$.

*Part 2*
Let $A$ be the set of marked boxes, then $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of $\{1, 2, \ldots, n\}$ with $1 \leq a_1 < a_2 \cdots < a_m \leq n$ and $0 < m \leq n$. A successful jump of the fleas can be associated with a bijection $\pi : \{1, \ldots, m\} \to A$ such that $i \leq \pi(i) \leq i + d$ for all $i$. The number of such successful jumps is equal to the permanent of the $(0, 1)$-matrix $C$, of size $m$ by $m$, defined by $c_{ij} = 1$ if and only if $i \leq a_j \leq i + d$. The probability that after the jump the $m$ marked boxes are occupied is per $(C)/(d+1)^m$.


*References*
[1] Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[2] The Dancing School Problems: http://www.jaapspies.nl/mathfiles/problems.html


**Problem 2006/2-C** We are given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ plus a sub-$\sigma$-algebra $\mathcal{C}$ of $\mathcal{A}$. We are also given a real-valued function $f$ on $X \times Y$ that is measurable with respect to the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$ (generated by the family $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$). Furthermore, each horizontal section $f_y$ is measurable on $X$ with respect to $\mathcal{C}$. Prove or disprove: $f$ is measurable with respect to $\mathcal{C} \otimes \mathcal{B}$.

**Solution** No solutions were sent in. The solution below is based on that of the proposer Klaas Pieter Hart.
Here is a counterexample. Let $X = Y = \omega_1$, the first uncountable ordinal. Let $\mathcal{A} = \mathcal{B} = \mathcal{P}(\omega_1)$, the power set. Let $\mathcal{C}$ be the $\sigma$-algebra that consists of the countable subsets of $\omega_1$ and those with countable complements.
The set $L = \{(\alpha, \beta) : \alpha \leq \beta < \omega_1\}$ belongs to $\mathcal{A} \otimes \mathcal{B}$ but not to $\mathcal{C} \otimes \mathcal{B}$. Its horizontal sections of $L$ are countable — they are of the form $[0, \beta]$ — and therefore they belong to $\mathcal{C}$. This means that the characteristic function of $L$ is a counterexample.

*$L$ belongs to $\mathcal{A} \otimes \mathcal{B}$*
To show that $L$ belongs to $\mathcal{A} \otimes \mathcal{B}$ we construct a sequence $\langle A_n \times B_n : n \in \mathbf{N} \rangle$ of rectangles such that $L = \bigcap_m \bigcup_{n \geq m} A_n \times B_n$. This is done quite indirectly.
By recursion we construct a sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ of infinite subsets of $\mathbf{N}$ with the following property: if $\beta < \alpha$ then $X_\alpha \setminus X_\beta$ is finite and $X_\beta \setminus X_\alpha$ is infinite — we abbreviate this as $X_\alpha \subset^* X_\beta$. For $\alpha < \omega_1$ we write $Y_\alpha = \mathbf{N} \setminus X_{\alpha+1}$ and we observe that the two sequences obtained in this way satisfy

$$\alpha \leq \beta \text{ if and only if } X_\alpha \cap Y_\beta \text{ is infinite.} \qquad (*)$$

From the $X_\alpha$ and $Y_\alpha$ we define $A_n = \{\alpha : n \in X_\alpha\}$ and $B_n = \{\alpha : n \in Y_\alpha\}$. The equivalence $(*)$ is translated in terms of the $A_n$ and $B_n$ reads

$$(\alpha, \beta) \in L \text{ if and only if } (\alpha, \beta) \in A_n \times B_n \text{ infinitely often.}$$

But this means $L = \bigcap_m \bigcup_{n \geq m} A_n \times B_n$, as promised.
It remains to construct the $X_\alpha$. Start with $X_0 = \mathbf{N}$. At successor stages let $X_{\alpha+1}$ be the even numbered elements of $X_\alpha$ (in its monotone enumeration). If $\alpha$ is a limit ordinal enumerate $\{\beta : \beta < \alpha\}$ in a simple sequence $\{\beta_n : n \in \mathbf{N}\}$ and recursively let $x_n$ be the minimum of $\bigcap_{i \leq n} X_{\beta_n} \setminus \{x_i : i < n\}$; then let $X_\alpha = \{x_n : n \in \mathbf{N}\}$.

*$L$ does not belong to $\mathcal{C} \otimes \mathcal{B}$*
Let $\mathcal{D}$ denote the family of those subsets $Z$ of $\omega_1 \times \omega_1$ for which one can find an ordinal $\alpha$ and a subset $A$ of $\omega_1$ such that $Z \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A$. We show that $\mathcal{D}$ is a $\sigma$-algebra that contains all rectangles $C \times B$ with $C \in \mathcal{C}$ and $B \in \mathcal{B}$; this implies $\mathcal{C} \otimes \mathcal{B} \subseteq \mathcal{D}$. Clearly $L$ does not belong to $\mathcal{D}$ and hence not to $\mathcal{C} \otimes \mathcal{B}$.

**$\mathcal{D}$ is a $\sigma$-algebra** Clearly $\omega_1 \times \omega_1$ belongs to $\mathcal{D}$.
If $Z$ belongs to $\mathcal{D}$ then so does its complement $Z^c$: if $Z \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A$ then $Z^c \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A^c$.
Let $\langle Z_n \rangle_n$ be a sequence of elements of $\mathcal{D}$, with associated ordinals $\alpha_n$ and sets $A_n$. Let $\alpha = \sup_n \alpha_n$ and $A = \bigcup_n A_n$; then $\bigcup_n Z_n \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A$, so $\bigcup_n Z_n$ belongs to $\mathcal{D}$.

**Every rectangle $C \times B$, with $C \in \mathcal{C}$ and $B \in \mathcal{B}$, belongs to $\mathcal{D}$** If $C$ is countable let $\alpha$ be such that $C \subseteq [0, \alpha)$ then $C \times B \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times \emptyset$.
If $C$ is co-countable let $\alpha$ be such that $C^c \subseteq [0, \alpha)$ then $C \times B \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times B$.

*Remarks*
In fact one has $\mathcal{D} = \mathcal{C} \otimes \mathcal{B}$. To see this let $Z \in \mathcal{D}$, with associated ordinal $\alpha$ and set $A$. Now observe that $Z$ is the union of $[\alpha, \omega_1) \times A$ and $\bigcup_{\beta < \alpha} \{\beta\} \times Z_\beta$, where $Z_\beta$ denotes $\{\gamma : (\beta, \gamma) \in Z\}$. This expresses $Z$ as a countable union of rectangles from $\mathcal{C} \otimes \mathcal{B}$.
In Kunen's thesis [1] it is shown that, in fact, $\mathcal{A} \otimes \mathcal{B}$ is the whole power set of $\omega_1 \times \omega_1$; the proof that $L$ belongs to the algebra is a simplification, for this special case, of Kunen's argument. The algebras $\mathcal{C}$ and $\mathcal{A}$ are quite far apart and rather extreme. I do not know what the answer is for more familiar $\sigma$-algebras. Specifically: can one prove that $f$ is $\mathcal{C} \otimes \mathcal{B}$-measurable when both $X$ and $Y$ both are the real line, $\mathcal{A}$ and $\mathcal{B}$ the $\sigma$-algebra of Lebesgue-measurable sets and $\mathcal{C}$ the $\sigma$-algebra of Borel sets?

*Reference*
[1] Kunen, Kenneth, *Inaccessibility properties of cardinals*, Ph.D. thesis, Stanford University, 1968.

**Results of Session 2006/2**

| | *Name* | A | B | C | D | *Total* |
|---|---|---|---|---|---|---|
| 1. | Brans | 8 | 0 | 0 | 0 | 24 |
| 1. | Vandendriessche | 8 | 0 | 0 | 0 | 24 |

**Final Table after Session 2006/2**
We give the top 3, the complete table can be found on the UWC website.

| | *Name* | *Points* |
|---|---|---|
| 1. | Arne Smeets | 28 |
| 2. | Annelies Horré | 21 |
| 3. | Ferry Kwakkel | 19 |