Problemen/UWC

Universitaire Wiskunde Competitie

De Universitaire Wiskunde Competitie (UWC) is een ladderwedstrijd voor studenten. De uitslagen worden tevens gepubliceerd op de internetpagina http://www.nieuwarchief.nl/uwc

Ieder nummer bevat de ladderopgaven A, B, en C waarvoor respectievelijk 30, 40 en 50 punten kunnen worden behaald. Daarnaast zijn er respectievelijk 6, 8 en 10 extra punten te winnen voor elegantie en generalisatie. Er worden drie editieprijzen toegekend, van 100, 50, en 25 euro. De puntentotalen van winnaars tellen voor 0, 50, en 75 procent mee in de laddercompetitie. De aanvoerder van de ladder ontvangt een prijs van 100 euro en begint daarna weer onderaan. Daarnaast wordt twee maal per jaar een ster-opgave aangeboden waarvan de redactie geen oplossing bekend is. Voor de eerst ontvangen correcte oplossing van deze ster-opgave wordt eveneens 100 euro toegekend.
Groepsinzendingen zijn toegestaan. Elektronische inzending in LaTeX wordt op prijs gesteld. De inzendtermijn voor de oplossingen sluit op 1 mei 2005. Voor een ster-opgave geldt een inzendtermijn van een jaar.
De Universitaire Wiskunde Competitie wordt gesponsord door Optiver Derivatives Trading en wordt tevens ondersteund door bijdragen van de Nederlandse Onderwijs Commissie voor Wiskunde en de Vereniging voor Studie- en Studentenbelangen te Delft.

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Problem A

Calculate
$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^3}.$$

Problem B

On a ruler of length 2 meter are placed 100 black ants and one red ant. Each ant walks with a speed of 1 meter/minute. If two ants meet then both turn 180^0. So does an ant that reaches the end of the ruler. At the start the red ant is exactly in the middle. Calculate the probability that the red ant is exactly in the middle after 4 minutes.

Problem C

We call a triangle integral if the sides of the triangle are integral. Consider the integral triangles with rational circumradius.

1. Prove that for any positive integral p there are only a finitely many integral q such that there exists an integral triangle with circumradius equal to p/q.
2. Prove that for any positive integral q there exist infinitely many integral triangles with circumradius equal to p/q for an integral p with $\gcd(p,q) = 1$.

Problem D

This problem has appeared earlier in round 2004/2. At that time no submissions were received. We reprint it here with a hint.
Quasiland has 30.045.015 inhabitants. Every two inhabitants are each others friend or foe. Any two friends have exactly one mutual friend and any two foes have at least ten mutual friends.

1. Describe the relations between the inhabitants.
2. Is it possible that less people live in Quasiland, while the inhabitants are still friend or foe as above?

Problem E$\star$

Alice and Bob play a game. Alice places $n-1$ candles on a square cake. Bob places an extra candle in the bottom-left corner. Then, for each candle, he cuts a rectangular piece of cake such that the candle is at the bottom-left corner and no other candles are in the rectangle. Bob gets all these n pieces of cake.

1. Is it always possible for Bob to get more than half of the cake?
2. What is the optimal strategy for Alice to hold onto as much cake as possible?

Edition 2004/2

Op de ronde 2004/2 van de Universitaire Wiskunde Competitie ontvingen we inzendingen van Filip Cools, Kenny De Commer, Annelies Horré en Jaap Spies.

Problem 2004/2-A

The sequence 333111333131333111333... is identical to the sequence of its block lengths. Compute the frequency of the number 3 in this sequence.

Solution This problem has been solved by Filip Cools, Kenny De Commer, Annelies Horré and Jaap Spies. Jaap Spies's solution is given here.

This sequence is known as the Kolakoski-(3,1) sequence. See N.J.A. Sloane's On-Line Encyclopedia of Integer Sequences, sequence number $A064353$, which is in fact the Kolakoski-(1,3) sequence, different only in the first position (see also [1]).

Michael Baake and Bernd Sing wrote: Unlike the (classical) Kolakoski sequence on the alphabet $\{1,2\}$, its analogue on $\{1,3\}$ can be related to a primitive substitution rule (see [2] and [3]). We base our calculations on section 2 of this paper.

Let $A = 33$, $B = 31$ and $C = 11$. In the case of $\mathrm{Kol}(3,1)$ the substitution σ and the matrix M of the substitution are given by

$$\sigma : \begin{array}{ccc} A & \mapsto & ABC \\ B & \mapsto & AB \\ C & \mapsto & B \end{array} \quad \text{and} \quad M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where $m_{ij} = 1$ if and only if there is corresponding mapping in σ, for instance $A \mapsto ABC$ corresponds to the fist column of M, etcetera.

An infinite fixed point can be obtained as follows:

$$A \mapsto ABC \mapsto ABCABB \mapsto \ldots$$

This corresponds to

$$333111333131\ldots$$

which is the unique infinite $\mathrm{Kol}(3,1)$. The matrix M is primitive because M^3 has only positive entries. The characteristic polynomial $P(\lambda)$ of M is

$$P(\lambda) = \lambda^3 - 2\,\lambda^2 - 1,$$

and has one real root λ_1 and two complex roots $\lambda_{2,3}$. We have

$$2.205569 \approx \lambda_1 > 1 > |\lambda_2| = |\lambda_3| \approx 0.67$$

According to the Perron-Frobenius Theorem there is a positive eigenvector to λ_1. We easily verify that $\mathbf{x_1} = (\lambda_1, \lambda_1^2 - \lambda, 1)^T$ is such an eigenvector.

Starting with $\mathbf{x}(0) = (1,0,0)^T$ we define

$$\mathbf{x}(k+1) = M\mathbf{x}(k)$$

The asymptotical behavior of this system will be of the form $\mathbf{x}(n) = c \cdot (\lambda_1)^n \mathbf{x_1}$ for some value of c.

From $\mathbf{x}(n)$ we can calculate the number of A's, B's and C's. In $A = 33$ there are two 3's, etcetera, so we can easily calculate the relative frequencies of the letters of the alphabet. The frequency of the '3':

$$\rho_3 = \frac{2 \cdot \lambda_1 + 1 \cdot (\lambda_1^2 - \lambda) + 0 \cdot 1}{2 \cdot (\lambda_1^2 + 1)} \approx 0.6027847150$$

References

[1] http://www.research.att.com/projects/OEIS?Anum=A064353
[2] Baake, Sing: Kolakoski-(3,1) is a (deformed) Model Set, Canad. Math. Bull. 47, No. 2, 168–190 (2004)
[3] http://arxiv.org/abs/math.MG/0206098

Problem 2004/2-C

Let A be a ring and let $B \subset A$ be a subring. As a subgroup, B has finite index in A. Show that there exists a two-sided ideal I of A such that $I \subset B$ and I has finite index as a subgroup of A.

Solution This problem has been solved by Filip Cools, Kenny De Commer and Jaap Spies. Jaap Spies's solution is given here. We have A and B as defined above. The index $[A : B] = k < \infty$ or with other words, the additive factor group A/B is a finite Abelian group build from co-sets of type $x + B$.
Let $E(G)$ be the ring of endomorphisms of the Abelian group G. We define a ring homomorphism $f\colon B \to E(A/B)$: for $a \in B$ we define $f\colon a \mapsto \alpha$ with $(x + B)\alpha = xa + B$. Note that we use here the *right* function notation, avoiding the notion of anti-homomorphism (see [2]).
The kernel of f is $L = \{a \in B | Aa \subset B\}$, L is the largest left-ideal of A with $L \subset B$. The factor group B/L is isomorphic to a subgroup of $E(A/B)$, so B/L is a finite Abelian group and since $(A/L)/(B/L) \cong A/B$ it follows that A/L is finite Abelian.
We now consider the ring homomorphism $g\colon A \to E(A/L)$: for $b \in A$ we define $g\colon b \mapsto \beta$ with $\beta(x + L) = bx + L$. Its restriction to L, $g_L\colon L \to E(A/L)$ has kernel $I = \{a \in L | aA \subset L\} = \{a \in B | Aa \subset B \wedge aA \subset B\}$.
I is the largest two-sided ideal of A with $I \subset B$. We have L/I finite and hence A/I is a finite Abelian group, so $[A : I] < \infty$.

References

[1] Marshall Hall, Jr. The Theory of Groups, Macmillan, New York, 1959.
[2] http://planetmath.org/encyclopedia/UnitalModule.html

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1.	Filip Cool	8		8	64
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Ladderstand Universitaire Wiskunde Competitie

We vermelden de top 3. Voor de complete ladderstand verwijzen we naar de UWC-website.

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1.	Tom Claeys	138
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