Problemen/UWC

Universitaire Wiskunde Competitie

De Universitaire Wiskunde Competitie (UWC) is een ladderwedstrijd voor studenten, georganiseerd in samenwerking met de Vlaamse Wiskunde Olympiade. De opgaven worden tevens gepubliceerd op de internetpagina http://academics.its.tudelft.nl/uwc

Ieder nummer bevat de ladderopgaven A, B, en C waarvoor respectievelijk 30, 40 en 50 punten kunnen worden behaald. Daarnaast zijn er respectievelijk 6, 8 en 10 extra punten te winnen voor elegantie en generalisatie. Er worden drie editieprijzen toegekend, van 100, 50, en 25 euro. De puntentotalen van winnaars tellen voor 0, 50, en 75 procent mee in de laddercompetitie. De aanvoerder van de ladder ontvangt een prijs van 100 euro en begint daarna weer onderaan. Daarnaast wordt twee maal per jaar een ster-opgave aangeboden waarvan de redactie geen oplossing bekend is. Voor de eerst ontvangen correcte oplossing van deze ster-opgave wordt eveneens 100 euro toegekend.
Groepsinzendingen zijn toegestaan. Elektronische inzending in LaTeX wordt op prijs gesteld. De inzendtermijn voor de oplossingen sluit op 1 augustus 2004. Voor een ster-opgave geldt een inzendtermijn van een jaar.
De Universitaire Wiskunde Competitie wordt gesponsord door Optiver Derivatives Trading en wordt tevens ondersteund door bijdragen van de Nederlandse Onderwijs Commissie voor Wiskunde en de Vereniging voor Studie- en Studentenbelangen te Delft.

Opgave A

The sequence 333111333131333111333...

is identical to the sequence of its block lengths. Compute the frequency of the number 3 in this sequence.

Opgave B

In Quasiland every two people are each others friend or foe. Any two friends have exactly one mutual friend and any two foes have at least ten mutual friends. Can there be foes in Quasiland?

Opgave C

Let A be a ring and let $B \subset A$ be a subring. As a subgroup, B has finite index in A. Show that there exists a two-sided ideal I of A such that $I \subset B$ and I has finite index as a subgroup of A.

Editie 2003/4

Voor de UWC ontvingen wij inzendingen van Syb Botma, Filip Cools en Joeri Vanderveken, Kenny De Commer, en Hendrik Hubrechts.

Opgave 2003/4-A

For each non-negative integer n, let a_n be the number of digits in the decimal expansion of 2^n that are at least 5. For example, $a_{16} = 4$ since $2^{16} = 65536$ has four digits that are 5 or higher. Evaluate the sum $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$.

Oplossing By Filip Cools and Joeri Vanderveken, Kenny De Commer, Hendrik Hubrechts, Bert Jagers, and Jaap Spies.
Here is the solution of Hendrik Hubrechts. In this solution $x \mod y \in \{0, 1, \ldots, y-1\}$. Let $M \in \mathbf{N}$. For $n \in \mathbf{N}$ define $c_n = 1$ if $2^n \mod 2M \geq M$ and $c_n = 0$ otherwise. We claim that $\sum_{n=1}^{\infty} \frac{c_n}{2^n} = \frac{1}{M}$.

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Using the identities

$$(2^n \mod 2M) - (2^n \mod M) = Mc_n\,, \quad \frac{2^n \mod 2M}{2} = 2^{n-1} \mod M$$

for $n \geq 1$, the claim can be deduced as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{c_n}{2^n} &= \frac{1}{M}\left(\sum_{n=0}^{\infty} \frac{2^n \mod 2M}{2^n} - \sum_{n=0}^{\infty} \frac{2^n \mod M}{2^n}\right)\\ &= \frac{1}{M}\left(1 + \sum_{n=1}^{\infty} \frac{2^{n-1} \mod M}{2^{n-1}} - \sum_{n=0}^{\infty} \frac{2^n \mod M}{2^n}\right) = \frac{1}{M}.\end{aligned}$$

It is possible to generalize this with k^n instead of 2^n and $c_n = j$ for the largest $j \in \{0, 1, \ldots, k-1\}$ such that $k^n \mod kM \geq jM$. The sum the still adds up to $\frac{1}{M}$. Further generalizations are possible, but the statements get more involved.
With the help of the above claim we can prove a generalized version of the problem. Let $N \geq 1$ and let a_n be the number of digits $\geq N$ in the expansion of 2^n in the $2N$-number system. Then

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} = \frac{2}{2N-1}.$$

To prove this, apply the claim with $M_i = N(2N)^i$ for $i \in \mathbf{N}$. The i-th digit in the expansion of 2^n in the $2N$-number system is $\geq N$ if and only if

$$2^n \mod (2N)^i \geq N(2N)^{i-1}$$

or equivalently, $2^n \mod 2M_{i-1} \geq M_{i-1}$. Hence the sum is equal to

$$\frac{1}{M_0} + \frac{1}{M_1} + \frac{1}{M_2} + \ldots = \frac{2N}{N(2N-1)}.$$

The case $N = 5$ shows that the answer to the problem is $\frac{2}{9}$.
This problem was suggested by Jan van de Lune.

Opgave 2003/4-B Let G be a group such that squares commute and cubes commute, i.e., $g^2h^2 = h^2g^2$ and $g^3h^3 = h^3g^3$ for all $g, h \in G$. Show that G is abelian.

Oplossing This problem was solved by Filip Cools and Joeri Vanderveken, Kenny De Commer, Bert Jagers, Nicky Hekster and Hendrik Hubrechts. Bert Jagers and Nicky Hekster have generalized the problem to commuting powers that are coprime.
Here is the solution of Hendrik Hubrechts. Define $\alpha = gh$ and $\beta = hg$. We will show in three steps that $\alpha = \beta$. First observe that

$$\alpha\beta\alpha = gh^2g^2h = gg^2h^2h = g^3h^3 = h^3g^3 = hg^2h^2g = \beta\alpha\beta$$

This implies that

$$(\alpha\beta\alpha)\alpha = \beta\alpha\beta\alpha = \beta(\alpha\beta)^2\beta^{-2}\beta = \beta^{-1}(\alpha\beta\alpha)\beta^2 = \alpha\beta^3$$

which implies that $\alpha^2 = \beta^2$. Now define $a = \alpha\beta$ and $b = \beta\alpha$ and apply the same calculation to find that $a^2 = b^2$, or, $\alpha\beta\alpha\beta = \beta\alpha\beta\alpha = \beta^2\alpha\beta$. Now $\beta\alpha\beta$ cancels so $\alpha = \beta$.
This problem was suggested by Hendrik Lenstra and Bart de Smit for coprime powers n and m, instead of 2 and 3.

Opgave 2003/4-C
Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$. Set $S_n = \sum_{k=1}^{n} X_k$. Calculate

$$P\{\exists n \geq 1 \text{ such that } S_{3n} = n\}.$$

Oplossing By Syb Botma, Filip Cools and Joeri Vanderveken, Hendrik Hubrechts, H. Reuvers and Jaap Spies.
Here is a slight generalization of the solution of Filip Cools and Joeri Vanderveken. Let $m \geq 2$ and denote $\sum_{n=1}^{\infty} P(S_{mn} = n)$ by A_m. Now,

$$\begin{aligned}
&P(\exists n \geq 1 : \; S_{mn} = n) \\
&= \sum_{n=1}^{\infty} P(S_{mn} = n \;\wedge\; \forall i \in \{1, \ldots n-1\} \colon S_{mi} \neq i) \\
&= A_m - \sum_{n=1}^{\infty} P(S_{mn} = n \;\wedge \exists i \in \{1, \ldots n-1\} \colon S_{mi} = i) \\
&= A_m - \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} P(S_{mn} = n \;\wedge\; S_{mi} = i \;\wedge\; \forall j \in \{i+1, \ldots n-1\} \colon S_{mj} \neq j) \\
&= A_m - \sum_{i=1}^{\infty} \sum_{n=i+1}^{\infty} P(S_{mi} = i) \cdot P(S_{m(n-i)} = n-i \;\wedge\; \forall j \in \{1, \ldots n-i-1\} \colon S_{mj} \neq j) \\
&= A_m - \sum_{i=1}^{\infty} P(S_{mi} = i) \cdot \sum_{n=i+1}^{\infty} P(S_{m(n-i)} = n-i \;\wedge\; \forall j \in \{1, \ldots n-i-1\} \colon S_{mj} \neq j) \\
&= A_m - \sum_{i=1}^{\infty} P(S_{mi} = i) \cdot \sum_{n=1}^{\infty} P(S_{mn} = n \;\wedge\; \forall j \in \{1, \ldots n-1\} \colon S_{mj} \neq j) \\
&= A_m \left(1 - \sum_{n=1}^{\infty} P(S_{mn} = n \;\wedge\; \forall j \in \{1, \ldots n-1\} \colon S_{mj} \neq j) \right) \\
&= A_m (1 - P(\exists n \geq 1 : \; S_{mn} = n)).
\end{aligned}$$

This implies that

$$P(\exists n \geq 1 : \; S_{mn} = n) = \frac{A_m}{1 + A_m}.$$

Now take $m = 3$ and observe that $S_{3n} = n$ if and only if $X_k = -1$ occurs exactly n times. Hence $A_3 = \sum_{n=1}^{\infty} \binom{3n}{n} \frac{1}{2^{3n}}$, which evaluates to $\frac{3}{\sqrt{5}}$ using Maple. Thus,

$$P(\exists n \geq 1 : \; S_{3n} = n) = \frac{3}{3 + \sqrt{5}}.$$

This problem was proposed by Mike Keane.

Uitslag Editie 2003/4
De weging van de opgaven is 3 : 4 : 5.

	Naam	A	B	C	*Totaal*
1.	Hendrik Hubrechts	9	9	9	108
2.	Filip Cools en Joeri Vanderveken	8	8	9	101
3.	Kenny De Commer	9	8	–	59
4.	Syb Botma	–	–	8	40

De top 5 in de laddercompetitie is als volgt:

	Naam	*Punten*
1.	Syb Botma	180
2.	Tom Claeys	138
3.	Gerben Stavenga et.al.	136
4.	Peter Bruin	99
5.	Kenny De Commer	90