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Mathematics as concept-mongering

Off the beaten track

Opvattingen over het leren zijn voortdurend aan veranderingen onderhevig. De in Nederland overheersende leertheorie of — zoals Freudenthal hier gezegd zou hebben — achtergrondfilosofie wordt vaak aangeduid met het woord constructivisme. Veranderingen in ons onderwijs zoals bijvoorbeeld de invoering van het zogenaamde studiehuis zijn vaak terecht of onterecht gemotiveerd door deze eigentijdse opvatting van leren. Deze ontwikkeling is al lang onomkeerbaar en heeft nogal concrete gevolgen voor ons allemaal. Maar wat behelst dit constructivisme eigenlijk en wat betekent het voor ons wiskundeonderwijs?

Stephan Hußmann, die in 2001 is gepromoveerd op een proefschrift over een leeromgeving die recht probeert te doen aan de opvattingen van het constructivisme, licht deze leertheorie en de mogelijke consequenties voor het wiskundeonderwijs toe. Sinds oktober 2003 is Hußmann hoogleraar wiskundedidactiek in Karlsruhe.

Wittenberg said doing mathematics is 'Denken in Begriffen' ('thinking in concepts'). But 'Denken in Begriffen' is nothing special. With every transaction of things in our world, these get a special meaning for us, in the form of concepts. We are perceivers and not irritable objects in the perceptible environment. Our actions are guided by propositionally contentful intentions with the aim of altering what is going on around us. We are not only behavers, who respond to a stimulus from outside without touch of sense. We use concepts for understanding and structuring what is going on around us. With every act of perception we differentiate the world in a part which carries a meaning for us and the other part which does not. In this sense, wisdom is concept-mongering (Brandom 2001, p. 8).

Therefore structuring the world is an individual act and meaning is attributed by the individual. Things, represented in the mind, do not have an 'objective reality'.

This understanding is based upon Kant's overcoming of the separation of mind and body. For Descartes concepts were created in accordance with the properties of things. The mind consequently consisted of representations of representable objects. To this end, the objects are analysed until a clear (*clara*) and distinctive (*distincta*) representation is reached in the mind. The criterion of quality is certainty. Kant contradicted this view by developing a prescriptive understanding of the usage of concepts. Conceptually structured activity distinguishes itself via its normative character (ibid, p. 9). "His [Kant's] fundamental insight is that judgements and actions are to be understood to begin within terms of the special way in which we are responsible for them" (ibid, p. 8). We are responsible for with what and how we furnish the mind with concepts. The criterion of quality for concepts is then the accuracy of their practical application. Thus, the mind is not dependent on the properties of things, but things depend on the applied concepts.

Now, when every human being perceives something different, attributes different meanings, how can it be that human beings can understand each other and come to similar conclusions from the observed phenomena? Possibly there is an ideal *a priori-field*, from which similar rules of thinking apply for every person (Leibnitz). This theory was contradicted by Hume who, for example, placed the concept of causality entirely in the habitual fields of human experience. But, even if causality is born out of experience, how is man at all able to combine and classify phenomena from his accessible environment. How do comparisons originate, how do differentiations originate? Doesn't man need general rules of thinking to reason reliably (for himself and others), and to bring the reasons for his behavioural pattern and attitude into a 'reasonable' order? This question cannot be answered here. Moreover, even when the rules of thinking 'merely' originate from individual fields of experience, they are, because of their abstractness, more reliable indicators of human communication than attributions of meaning to things.

Beside the 'objective' rules of thinking, the processes of negotiating in the socially and culturally shared field of experience also contribute to a convergence of different concepts. Concepts are, so to speak, socially consolidated meanings manifested in linguistic terms. In this respect the question whether things can be represented in the mind as if things have an 'objective reality' is not the focus of our discussion. The aim is to understand ourselves as concept-users and as attributers of meaning (as if things have a 'subjective reality') (Brandom 2001, p. 7).

Experience is the index of meaning. The development of concepts has the aim of enabling the individual to manage the demands of everyday life. This is the source of any attribution of meaning. But every situation is *sui generis*. To overcome difficulties of everyday life situations without having to develop all strategies and concepts anew, we need general concepts which are independent of the concrete situation. The construction of such a concept network rests on experience and is carried out through an abstracting process from the concrete and situational experiences up to the abstracted concepts, with the goal of maintaining the *autopoeisis* of the system, or, biologically expressed, safeguarding the system's ability to survive.

Mathematics as concept-thinking — a subject-specific perspective

Mathematics is a science which, in a special way, prescinds concepts from the concrete, with the aim of developing a precise and definite concept network, and tracks down the rules of thinking. Against this background, Wittenberg's statement that mathematics is 'thinking in concepts' is understandable. With mathematics, it is not only possible to structure phenomena in our world with concepts, but also to experience abstracted mathematical objects as 'a deductive regulated world of its own' (Winter 1995). This becomes apparent especially in two characteristic aspects of mathematics that day-to-day concept development generally does not possess: the precision of the mathematical concepts and the validity of the arguments.

The everyday ability to react is also possible with non-precise concepts. That is really precisely their strength, because a reduction of the meaning of words used in everyday language would imply an exponential rise in necessary new words. The precision, conciseness, and shortness of the language of mathematics contrast with the redundancies and possibilities of an anticipating and hermeneutic understanding of everyday language. Experience with and understanding of the world are hence always *concept-mongering* and mathematics as a science of abstract objects can be described as a special form thereof, as *concept-thinking*.

What does this mean for teaching?

In school mathematics, the perspective of the student, which is moulded by the concrete and comprehensible, is confronted with idealisations and abstractions of scientific mathematics which is detached from this concrete reality. In many cases, the result is the mathematical concept coagulating to algorithms. Not the conceptual 'mongering' with the concept of integral calculus or fractional arithmetic is learned, through which mathematics can be experienced as a deductive orderly world of its own, but the routine processing of derivative rules dominates as primal characteristic in the field of vision onto a much richer and more comprehensive concept. The concrete is sought after in formalised calculating rules, instead of admitting the concreteness of the phenomena in the proper place.

Hence, mathematics in school manoeuvres in a field, which is determined by at least three areas of conflict, that is, those between

- concretising and abstracting,
- singular and consolidated concepts,
- construction and instruction.

To a certain extent, these areas of conflict interact intensively. The last aspect concerns, in *epistemological view*, the issue of subject-dependent existence of the things reflected in our mind and, therefore, the issue of objectivism or constructivism. Constructivism is currently enjoying popularity as a 'new theory' in education. The traces of its roots go back to Xenophanes. He studied the question whether we can describe the 'world-in-itself'. But our knowledge of the world is derived from our experience. Therefore we have no way of checking the truth of our knowledge with the world, because every access to the world involves our experience. Kant (1787) suggested that the concepts of space and time were the necessary forms of human experience, rather than characteristics of the world. This implies that we cannot imagine what the structure of the real world might be like, only what the structure of our cognition is. Our ideas adjust to their practical application. This thought is, in biological respect, close to the concept of assimilation in Darwin's evolution theory: only *those* life-forms can survive which are capable of adapting to the given environmental conditions. This does not mean survival of the fittest; instead, adaptation can take place in many different ways. Piaget carried this theory over to the field of cognition and explained knowledge to be tied to target-orientated acting, with the knowledge of an object being its assimilation and accommodation in one's own knowledge structures (Piaget, 1977, p. 74). The objective is to achieve a balance between experienced reality (substantiveness) and cognitive structures. The quality of learning and knowledge is thus not determined by the quality of mapping of reality, but by the function in reality. The nature of learning is hence explicitly instrumental (von Glasersfeld, 1995): 1. "Knowledge is not passively received either through the senses or by way of communication"; 2. "Knowledge is actively built up by the cognizing subject"; 3. "The function of cognition is adaptive in the biological sense of the term, tending towards fit and viability"; 4. "Cognition serves the subject's organization of the experiental world, not the discovery of an objective ontological reality." (von Glasersfeld, 1995, p. 51).

In didactic-methodical respect the relationship of self-regulated learning and guidance provided by the teacher in mathematics lessons finds consideration in the third field of conflict. The following metaphor is to explain this and other actions of the three areas of conflict in their didactic-methodical dimensions.

Walks through the woods

Imagine that mathematics lessons had the task of making a certain closed-in area of woodland passable for the students. The elements of the wood present the mathematical content and mathematical phenomena of a well-defined mathematical area. The ways through the wood are the different ways through the content area and thereby also describe different modes of dealing with mathematics. In the wooded areas there exist well-trodden ways, on which passage is relatively easy and no special techniques have to be


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applied. However, on the small paths, which wind their way through the wood, one has to be skillful or seek knowledge for one's wanderings.

The different accesses and passages through the wood can be reduced to two main types. On the one side there is the guided tour on well-developed tracks and on the other side there is the individual access, maybe even without the usage of paths. The diagram indicates these two possibilities.

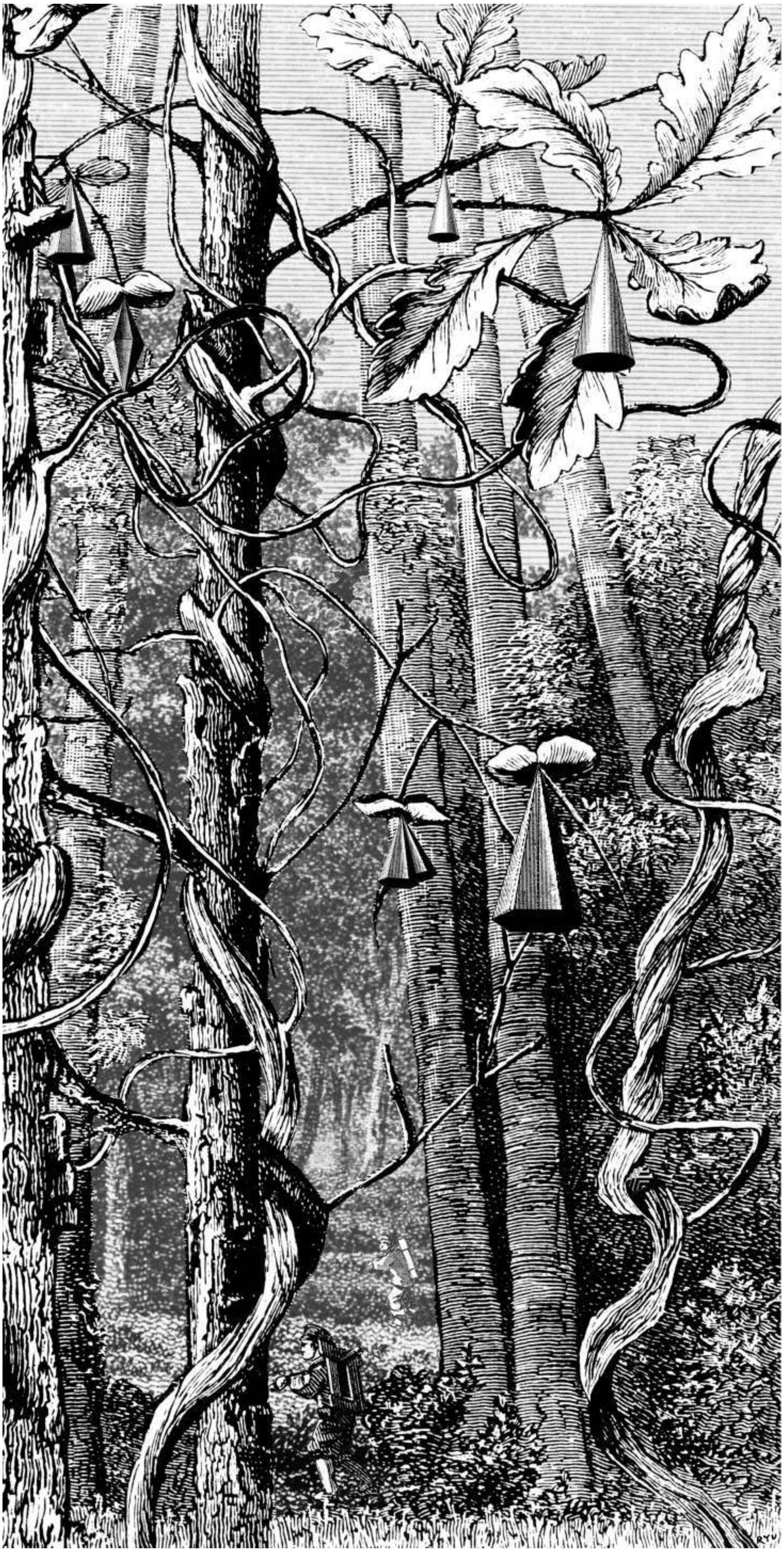
Access under guidance (of the teacher) has the advantage that the directional learning of the students is prepared in such a way that they are lead past the special attractions of the area. The ways are well constructed, so that the progression is not hindered by unnecessary barriers and the area can be explored unhindered. Care is taken to ensure that the distances covered do not over-tax the learners, that is, they are reasonably short and easy ones are covered before more difficult ones are tackled. The knowledge-controlling orientation is the deductive structure of mathematics. But guided tours do not only have advantages. Foreign control conduces all too quickly to no more paying attention to the way. So there is the danger that one loses the orientation, as a result of which the overall context remains inaccessible. The hiker will notice this at the latest when he is supposed to or wants to walk down the way again on his own. The short distances — easy ones first and then more difficult ones, without any considerable barriers — do not enable the hiker to also make use of the acquired competence in other situations, for example, if he hikes in 'rough' landscapes.

An alternative is offered by reconnaissance on own paths. The originality of the wood itself demands its reconnaissance. Not the well-developed ways, but tracks and own firebreaks step by step formulate the groundwork for the knowledge of the wood. In one stroke it is just the barriers which characterise the forest area in its speciality. Every overcome hindrance and every aha experience creates a tie between learners and mathematics — something a pre-structured way could never manage. The phenomena can be selected and examined in detail. But also here the hiker can lose the orientation: barriers can turn out to be too large and dead-end roads too long, and the hiker might overestimate his abilities to find a way through the bushes. Here, too, a teacher is needed, not in the role of a guide but in the function of an adviser. Depending on the specific situation of the individual student, he has to determine which support he or she requires to draw up new perspectives when one errs; he has to confirm that the way is suitable and promising for the student. And he has to show the learners the area as a whole to ensure they do not get lost on their own ways.

While one can describe the first approach as *routine-orientated concept-mongering*, the second approach is characterised through *problem-orientated concept-mongering*. Both activities encompass concrete and situationally placed objects of the experience world. While in the first case the objects could be unambiguously allocated to the consolidated concepts through demonstration and the attributions of meaning through the learner play a secondary role only, the learners have to monger with concepts to actually master the second approach. They themselves have to ask questions to define their singular access, make differentiations, develop processes, construct meanings and develop concepts based on their subjective experience field. The experience of individually steered concept forming strengthens the consolidated meaning which was obtained in the social negotiation process with the individually attributed meanings.

The process of concept-thinking as it is presented above begins with the reflection of the concept building process. In this process, similar structures in individual networks are compared with one another. The concrete problem situations are taken away and the used concepts in these situations are compared for structural similarity, so that an abstract concept, which is detached from the concrete, can emerge. To remain in our picture: step by step, the concrete wood was transposed into a map. The concrete tree, that is to say, the phenomenon containing the mathematics, still only appears as a symbol. Similar trees are given the same symbol. Whole wood segments are represented by higher-ranking symbols. The focus is only on the discovered mathematical phenomena and on the road and path network, the concept network that connects the phenomena with each other. This is described as *structure-orientated concept-mongering*.

Subsequently or simultaneously, the individual maps are compared, with the goal of checking the individual constructions for their suitability in the social context. From this comparison consolidated concepts originate in the school class, which generally stand up to comparison with the social consolidated concepts. That is, they can be matched with them, as concept development is subject to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the same rules of thinking and the concepts are related to similar contexts of experience.

How can this be accomplished in mathematics lessons?

What would learning arrangements look like, in which the students actually go their own way in learning mathematics? How can an area of woodland be enclosed suitably for students, so that, simultaneously, the intended mathematics is acquired as new knowledge at the end of the learning process? How can students be motivated, to actually think in concepts? As an example of a possible teaching/learning-environment, let us present a problem situation from the project *Discovering and Researching Mathematics — constructive learning with intentional problems* (cf. Hußmann 2002, 2003). Starting point and basis for these teaching/learning arrangements are *Intentional Problems*. These are complex, open, internal and external mathematical problem situations, which open the access to a theme area and should carry on far into the theme. The concepts and methods necessary for problem solving are developed by the students on their own and woven into a mathematical theory. For this purpose, the problems are conceived such that they require the discovery or invention of fundamental concepts of a theme area. In this process, the concept terms originate from the interplay of the reference context of the problem situation and the models developed by the learners to attribute meaning to the situation.

A very meaningful property of intentional problems is their difference in structure. Intentional problems are not a consecutive row of interesting problems for accessing a mathematical theme area, but they are related to one another in view of the central concept of the content area. As single problems they show the different facets of a concept or indicate special aspects. Overcoming the difference in structure, that is, the abstraction from the concrete through recognising the similarities and differences, leads the students to generally acceptable concepts.

An example of an access to integral calculus is given by the following problem situation (This is one of three problem situations (Hußmann 2003)).

Ms. Grat finds herself on the way from Munich into the Ruhr area with her lorry. At a routine motorway police control point her tachograph is checked. It is found out, that there is no trace between 8:00 a.m. and 9:00 a.m. When questioned Ms. Grat says that she had a long break at a motorway service station.

To begin the problem-orientated phase, the students formulate their own questions about the problem and are confronted with their individual access to the problem situation:

- Did Ms. Grat lie?
- How can the distance covered be determined?

To solve the problem the students need the tool of integral calculus, which, however they do not yet have at their disposal at this point in time. Consequently they have to develop the necessary mathematics themselves and they apply their already acquired skills. If the students hereby discover knowledge gaps, they will themselves recognise the necessity to fill these gaps individually. Because of this, the problem situation should contain enough knowledge already acquired to ensure that the learners can tie up to their previous knowledge and enough unknown to ensure that new knowledge can be built up. In the example, the students activate their previous knowledge, in which they approach the area under the curve with rectangles, trapezoids and triangles. They develop new concepts, for example by making use of still more and smaller areas, to be able to determine the distance as a whole. For this purpose, lower, upper, right or left sums, or more general Riemann sums, can be used. The problem-orientated phase is brought to an end, when the problem is solved.

Upon completion of the problem-orientated phase, the structure-orientated phase is initiated. Own approaches to the solution are considered and compared with other ways. Making reference to the work done on other problems, it is possible to recognise similar strategies which are independent of the specific problem situation, for example:

- the solution can be generated through cumulation of different products or areas;
- the result is more exact if more subproducts are used;
- the result thereby is invariant, irrespective of the choice of geometric figures or the type of totals formation.

This example makes quite clear how the mongering with concepts abstracts from the concrete situation. As learners proceed, mathematical objects themselves will be used as reference context and concept-mongering will be specified to concept-thinking.

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