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Johann Bernoulli lecture

The Wizards of Wall Street: did mathematics change finance?

Dit is een verkorte versie van de Johann Bernoulli-lezing uitgesproken door Paul Embrechts, hoogleraar verzekeringswiskunde aan de Eidgenössische Technische Hochschule in Zürich, op 21 mei 2002 in de Aula van het Academiegebouw van de universiteit in Groningen.

What field of science would one be talking about when hearing the following words: butterflies, eagles, rainbows, passports, Europeans, Americans, Asians, Parisians, Russians, knock-in, knock-out, barriers, swaps, swaptions, calls, puts, baskets, digitals, swings, down-and-under, ...? This rather strange zoology of terms comes from the realm of modern finance. All the words above refer to so-called financial derivatives. Most of them are options. A derivative is a financial product written on (derived from) another financial product. The latter is typically referred to as the underlying. A typical example is a so-called European call option written on a stock. This product (derivative) gives the holder the right (not the obligation) to exchange for the underlying stock at a predetermined date and price. The date is referred to as maturity, the price as strike. The buyer pays the seller a premium for this right. For instance, suppose a bank on March 28, 2002 writes a European call with maturity 1 year and strike 160.– CHF on Swiss Re N, then it promises the buyer (holder) of the call to deliver him one year from March 28, 2002 one stock Swiss Re N at the agreed price of 160.– CHF. A key question now concerns the calculation of a premium, fair for both buyer as well as seller. Swiss Re N closed on March 28 at 154.75 CHF. The *Wizards of Wall Street* mentioned in the title of this paper not only solved this problem, but also devised for the seller a perfect hedge; starting with the initial premium, they constructed a dynamic portfolio which allows the seller to exactly replicate the value of the call at maturity.

In this paper, I will discuss some mathematical techniques used in solving the above problem. Special attention will be given to the conditions underlying the solution. The world of derivatives will be placed in its historic context. Besides a brief excursion into the realm of risk management, I will also make some comments on current and future research in the field.

Pricing a European call

Returning to the example above, denote by S_t the stockprice at time t ($t = 0$ is today), T stands for maturity and K denotes the strike, then, at maturity, the value of the European call is

$$C(T) = (S_T - K)_+ = \max(S_T - K, 0)\,. \tag{1}$$

Given a risk free interest rate $r > 0$ in the market, a first intuitive guess of today's value of the claim $(S_T - K)_+$ at future time T is

$$C(0) = E\left(e^{-rT}(S_T - K)_+\right). \tag{2}$$

Here $E(X)$ denotes mathematical expectation of the random variable X defined on some basic probability space (Ω, F, P) where P stands for the (so-called physical) probability measure,

$$E(X) = \int_\Omega X(\omega)dP(\omega).$$

In order to make the latter point clear, I could have denoted $E^P(X) = E(X)$. It now turns out that (2) yields the wrong price (compare with (6))! A more intricate (so-called no-arbitrage) argument starting for instance from (6) yields the 'correct' price:

$$S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{3}$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)\,T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Moreover,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt, \tag{4}$$

the (cumulative) standard normal distribution function. The final parameter remaining unexplained is σ. The latter stands for the standard deviation (volatility) of the underlying stock. A more precise, mathematical definition will be given later, after the introduction of the model (10).

That the normal distribution function (4) has something to do with finance was early on clear to the Deutsche Bundesbank; see their old 10.– DM bank note with Karl Friedrich Gauss on it together with $\Phi(z)$ also referred to as the Gauss distribution. That Φ enters fundamentally in the pricing of derivatives, we owe to Fisher Black, Myron Scholes [4] and Robert C. Merton [22] and indeed very much depends on the conditions in the underlying model.

Formula (3) can without doubt be referred to as 'A Nobel formula', as indeed the 1997 Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel was given to Merton and Scholes (Black died some years earlier) "for a new method to determine the value of derivatives". As Keith Devlin wrote in 1997: "The award of a Nobel Prize to Scholes and Merton shows that the entire world now recognizes the significant effect on our lives that has been wrought by the discovery of that one mathematical formula." Little did Devlin realise how true his statement would become one year later.

There exist several ways to arrive at the Black-Scholes (-Merton) formula (3).

Solution 1. (PDE approach, Black and Scholes [4], Merton [22])
Denote the option price at time t $(0 \leq t \leq T)$ by $C(t) = f(t, S_t)$, then $f(t,s)$ satisfies the so-called Black-Scholes partial differential equation

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf \\ f(s,T) = (s-K)_+. \end{cases} \tag{5}$$

It is not difficult to show that (5) can be transformed into the heat equation and as such corresponds to one of the fundamental PDEs in physics. The reason for this will be given later.

Solution 2. (Martingale approach, Harrison and Kreps [15]; see Harrison and Pliska [16] for details on the early work)
So far we have only given one basic σ-algebra F of (measurable) events on Ω. In order to fully model financial markets, one has to introduce the notion of information. This is done through turning (Ω, F, P) into a filtered probability space $\left(\Omega, F, (F_t)_{t\geq 0}, P\right)$ where the σ-algebras F_t are increasing, i.e., for all $s \leq t$, $F_s \subset F_t$. The σ-algebra F_t can be viewed as the market information available at time t. One often takes the natural filtration $F_t = \sigma(S_s, s \leq t)$ generated by the underlying price process (S_t). This choice is however less appropriate when one wants to model for instance insider information; in that case one may want to augment F_t above by some extra σ-algebra G denoting the extra information available. This then leads to interesting mathematical problems for properties of stochastic processes under augmented filtrations; see for instance Amendinger et al. [1]. Given now a filtration, one can speak about the conditional expectation of X given all information up to and including time t, $E(X \mid F_t)$. The necessary correction to the intuitive formula (2) now becomes:

$$C(0) = E^Q\left(e^{-rT}(S_T - K)_+\right) \tag{6}$$

or indeed more generally for the value $C(t)$ of the call at time $t \in [0,T]$:

$$C(t) = E^Q\left(e^{-r(T-t)}(S_T - K)_+ \mid F_t\right). \tag{7}$$

In (6) and (7), expectation with respect to the physical measure is replaced by expectation with respect to a so-called risk-neutral probability measure Q. Again, at this point, some fundamental conditions on the underlying model for (S_t) enter.

Solution 3. (Binomial tree pricing, Cox, Ross and Rubinstein [5])
Whereas Solutions 1 and 2 presuppose a continuous time model for (S_t), Cox, Ross and Rubinstein came up with a discrete time solution which methodologically stands to the previous approaches as a random walk relates to its weak limit, Brownian motion (and hence the normal distribution). Assume that price processes can only go up or down with some (physical) probability p, $1-p$ respectively, i.e. for $t \in \mathbf{N}$,

$$S_{t+1} = \begin{cases} uS_t & \text{with probability } p \\ dS_t & \text{with probability } 1-p \end{cases} \tag{8}$$

where $0 < d < 1 + r < u$. Then for $\tau = T - t$, time to maturity,

$$C(t) = (1+r)^{-\tau} \sum_{j=0}^{r} \binom{r}{j} p^{*j} (1-p^*)^{r-j} \left(S_t u^j d^{\tau-j} - K\right)_+ \tag{9}$$

where

$$p^* = \frac{1+r-d}{u-d} \in (0,1).$$

Formula (9) is of the form (6) where Q corresponds to the binomial probabilities based on p^*. In general $p^* \neq p$, hence in (8) it is not important with which probability p prices go up or down. It is the stock's volatility (implicitly present in u and d) that plays the key role.

The above solutions are mathematically linked. The basic assumption on the underlying stock process (S_t) in Solutions 1 and 2 is geometric Brownian motion:

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{10}$$

where μ is a drift parameter, σ the (assumed constant!) volatility and (W_t) stands for standard Brownian motion. The fact that the increments of Brownian motion are normally distributed,

$W_t - W_s \sim N(0, t-s)$, explains, after some calculations, the Φ in (3). Indeed, in micro-economic theory interacting agents (traders) are viewed as gas particles bombarding each other (or better said, the price of the stock). As Brownian motion has almost surely nowhere differentiable sample paths, the so-called Itô stochastic differential equation (10) has to be defined in a proper way. The solution to (10) is called a diffusion, properties of which can be studied via the theory of PDEs, leading to Solution 1, or via the (equivalent) theory of stochastic (martingale) calculus, leading to Solution 2. Solution 3 can be seen as a discrete time version of Solution 2, indeed, Solution 2 can be obtained through a central limit argument (weak convergence) of Solution 3. Also note the absence of μ (in (10)) from the formula (3).

I have refrained from giving detailed references to the results above. By now, a multitude of textbooks exists on the subject. For a mathematician, a good place to start is Bingham and Kiesel [3] for a fairly easy introduction. Mathematically more demanding are for instance Musiela and Rutkowski [23] and Karatzas and Shreve [18]. These texts contain numerous references for further reading. An excellent introduction in discrete time is Föllmer and Schied [12].

What about the conditions

There is no doubt that the Black-Scholes-Merton formula (3), and more importantly the methodology developed for the rational pricing and hedging of financial derivatives, changed finance. As such, the (mathematical) Wizards of Wall Street had a non-trivial impact on the developments of financial markets over the last couple of decades. Not only did the new option pricing formula (3) work, it transformed the market. When the Chicago Options Exchange first opened in 1973, less than thousand options were traded on the first day. By 1995, over a million options were changing hands each day. So great was the role played by the Black-Scholes-Merton formula in the growth of the new options market that, when the American stock market crashed in 1978, the influential business magazine Forbes put the blame squarely onto that one formula. Scholes himself has said that it was not so much the formula that was to blame, but rather that market traders had not grown sufficiently sophisticated in how to use it.

However, much more important it is to realise under what assumptions (mathematically as well as economically) does the formula hold. Already Black said that he found it difficult to apprehend that a formula like (3) based on so many unrealistic assumptions was so widely used and did so well. Here is a partial list of the key underlying assumptions:

- constant volatility
- independent, normally distributed relative returns
- no-arbitrage
- self-financing strategies
- no frictions (taxes, dividends, transaction costs)
- infinite liquidity
- stocks tradable at every fraction
- efficient, rational, complete markets
- perfect hedging.

One can show (partly statistically) that all of the above assumptions are violated to some extend in practice. For several of them (for instance constant volatility and frictions) the theory can be salvaged and necessary adjustments to (3) be made. In the end however, there always remain conditions that may not hold (even approximately) for real markets. The 'may not' case typically occurs when markets are under stress, like the events surrounding the LTCM disaster in September 1998.

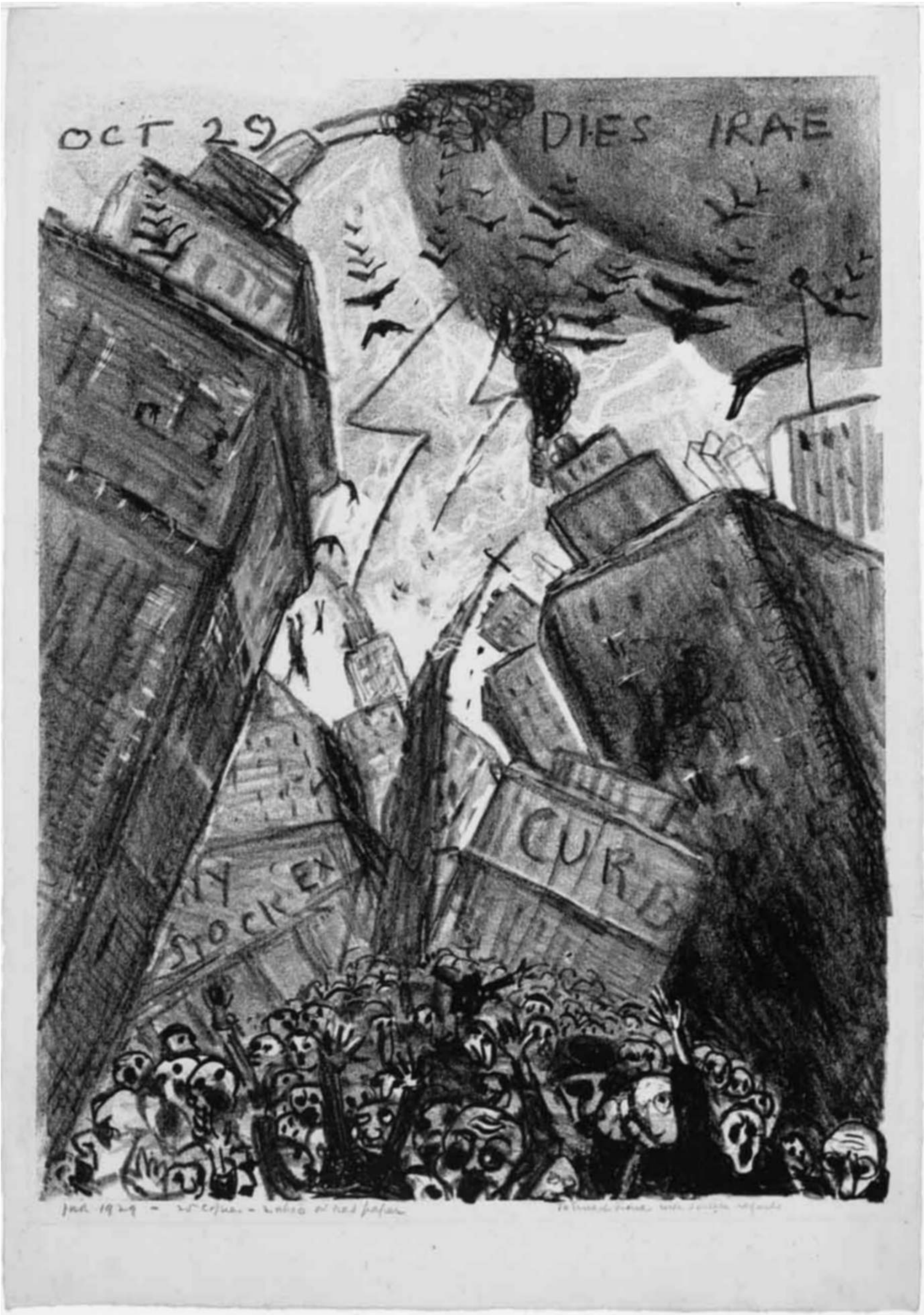
Long-Term Capital Management (LTCM) was a hedge fund set up around the former Salomon Brothers trader John Meriwether. With Merton and Scholes on the company's board, LTCM was using highly quantitative techniques for taking advantage (through leverage) of, according to their methodology, mispriced products. For those (relatively few but big) investors allowed to join, a money machine seemed to emerge. A dollar invested in the fund around March '94 grew as follows: 3/94 ($1), 3/95 ($1.40), 3/96 ($2.30), 3/97 ($3.50), 3/98 ($4), just short of its peak of around $4.10. July '98 was down to $3.50 before the lightening crash taking the fund down to about $0.30 by early September 1998. By then, the fund reached a complete collapse and was saved from bankruptcy by a (still hotly debated) deal set up by the New York Fed and several large international banks. The latter deal was made out of fear for a worldwide financial meltdown. A 'too big to fail' situation surrounded LTCM in those crucial days. I am not saying that (3) was to blame for this; no doubt however, a far too optimistic view on the robustness for the methodology underlying (3) had an important role to play in the fall of LTCM. Readers interested in the more detailed non-technical story can read Dunbar [10] or Lowenstein [21]. For an excellent, more technical discussion on which conditions mainly caused the bad performance of LTCM's risk management system as an early warning system, see Jorion [17].

In the aftermath of the 1998 LTCM (and other) disaster(s), the public transformed the *hailed* Wizards of Wall Street into the *failed* Wizards of Wall Street. My claim however is that not less, but *more* mathematical (critical) thinking is strongly needed. Mathematicians working in the field of finance (and insurance) have to communicate more forcefully the weaknesses/shortcomings and the assumptions underlying the models used. And let us not forget: mathematicians working in this area with a claim to applied relevance of their work *will have to study and understand the underlying economics*!

Some pricing techniques

By far the most useful economic device in the field of quantitative finance is the notion of no-arbitrage. An arbitrage opportunity is a self-financing strategy with zero initial value, which produces a non-negative final value with probability one and has a positive probability of a positive final value. By not allowing such strategies, economists can easily price new products exploiting their relationship with other existing ones. One example are the so-called currency triangles, as there is (US$/€, €/£, £/US$): these exchange rates must be perfectly linked, otherwise one could make a sure, riskless gain. A further example is the so-called put-call

Right: Oct 29 *Dies Irae*, 1929, James N. Rosenberg, litho, druk: George Miller. James Rosenberg was born in Pittsburgh, Pennsylvania, and grew up in New York City, attending Columbia University and graduating from Columbia Law School in 1898. Even after becoming a successful bankruptcy lawyer in Manhattan, Rosenberg nevertheless continued to cultivate a passion for art, which led to his becoming a collector. He painted and made prints in his spare time, discovering lithography under George Miller's tutelage in 1919 (source: *Life of the people*, Washington, Library of Congress.) Copyright: Estate of James N. Rosenberg, permission granted by Anne Geismar.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parity. A European put is the right to sell a given stock at a given date for a given price, hence for the buyer it has a value at maturity of

$$P(T) = (K - S_T)_+ \,. \tag{11}$$

Recalling from the first section the value of a call at time $0 \leq t \leq T$, $C(t)$ and denoting similarly by $P(t)$ the value of the corresponding put, then the put-call parity becomes, for $0 \leq t \leq T$,

$$S_t + P(t) - C(t) = Ke^{-r(T-t)}. \tag{12}$$

The easiest way of proving (12) is by assuming strict inequality for some $0 \leq t \leq T$ and then come up with a portfolio which shows a riskless profit by time $t = T$. Note that by the definitions of a European call and put ((1), (11)) one immediately has at maturity $t = T$, that

$$S_T + P(T) - C(T) = K. \tag{13}$$

An excellent reference on the use of arbitrage arguments in order to prove relationships like (12) is Cox and Rubinstein [6].

The early, main contribution of mathematics to finance is no doubt the formulation of a methodological foundation to the above economic no-arbitrage argument. The pricing equation (7) holds for general contingent claims Y (meaning $Y \in L^1(\Omega, F_T, P)$):

$$V_Y(t) = E^Q\left(e^{-r(T-t)}Y \mid F_t\right), \tag{14}$$

for $0 \leq t \leq T$ where V_Y denotes the value (or price) function of the claim Y. Applying now (14) to (13) and using a similar formula to (7) for a put yields:

$$E^Q\left(e^{-r(T-t)}S_T \mid F_t\right) + P(t) - C(t) = Ke^{-r(T-t)}.$$

At this point, a key observation has to be made, namely, for $0 \leq t \leq T$:

$$S_t = E^Q\left(e^{-r(T-t)}S_T \mid F_t\right)$$

or equivalently:

$$e^{-rt}S_t = E^Q\left(e^{-rT}S_T \mid F_t\right), \quad 0 \leq t \leq T. \tag{15}$$

This means that the discounted price process $\left(e^{-rt}S_t\right)$ is a (Q, F_t)-martingale. It is this fundamental link between no-arbitrage for the price process and its martingale property (15) which lies at the heart of the importance of modern stochastic calculus for mathematical finance. This link, starting with Harrison and Kreps [15], found its culmination point in the so-called Fundamental Theorem of Asset Pricing as discussed in Delbaen and Schachermayer [9]. It is fair to say that early on, economists were able to derive pricing formulae for derivatives using the very powerful (and intuitive) device of no-arbitrage. By showing that the no-arbitrage concept is 'equivalent' with a martingale property of the underlying discounted price process, the doors were opened for the analysis of much more complex (so-called exotic) options. A whole stochastic calculus industry for finance emerged. It still largely is a matter of taste to use stochastic calculus (martingale) techniques directly or go via the equivalent PDE theory. In order to get an idea on what type of options can be priced in practice, see for instance Lipton [20].

Above we saw that mathematics enters very fundamentally in order to establish a coherent methodology for the rational pricing of contingent claims. This is however only the beginning, the theory has been extended in a variety of ways. Some of these extensions are fundamental for practice, as for instance the analysis of pricing and hedging in incomplete markets. Other extensions contribute to a beautiful mathematical theory but offer little (if any) to the solution of real problems in finance; I refrain at this point from discussing examples of the latter category. Returning to the former, incompleteness of financial markets is of fundamental importance and is more the rule rather than the exception. A typical example of an incomplete market is one where jumps in the price process with random size occur. Contingent claims cannot be perfectly hedged (replicated), there are infinitely many equivalent (pricing) martingale measures Q and consequently, investors will have to indicate their attitude to risk. Notions like utility pricing and non-perfect hedging enter. Most of the modern textbooks on finance contain excellent accounts of the (non-trivial) mathematical theory. A very readable review paper is Schweizer [26]. A more in depth discussion on the use of mathematical techniques in finance is for instance to be found in Schachermayer [25].

As so oft in modern applied probability, and indeed as shown above very much so in mathematical finance, solving a practical problem posed can essentially be reduced to 'spot the martingale'! In view of the ever occuring ups and (especially) downs of financal markets, one may recall in this context the words of that famous gambler Giacomo Casanova (Venice, 1754): "At this same time I was being ruined at cards. Playing by the *martingale*, I lost very large sums; urged on by M.M., I sold all her diamonds, leaving her in possession of only five hundred zecchini. There was no more question of an elopement." This story leads us nicely to the next section, putting the above sketched development in a wider historical perspective.

Is history repeating itself

In the Code of Hammurabi, 1800 BC, the following text can be found "If any one owe a debt for a loan, or the harvest fail, or the grain does not grow for lack of water; in that year he need not give his creditor any grain, he washes his debt-tablet in water and pays no rent for this year." As is explained in Dunbar [10] p. 25, the above can be viewed as the debtors having an option to call upon the lendors to cover their interest payments in the event of crop failure, which effectively put a cap on their grain price exposure.

Hence derivatives in general, and options more in particular are not so new. All too often they are viewed as inventions of the 'capitalistic devil' and mathematicians seriously working on them ought to be scorned. I take a completely different view; financial options are so much part of every day life that it is an absolute necessity for mathematicians to take a serious interest. Who has not yet considered a prepayment option in a mortgage or a change from a fixed interest rate agreement to a variable one, or vice versa (a so-called swap). I go along with Steinherr [27] who claims in his excellent (pre LTCM) book that the development of derivatives markets, and the from this development established quantitative risk management tools, constitute no doubt one of the key innovations of the 20th Century. The main reason why the general public occasionally loaths these modern tools of finance is through their perceived triggering effects in crashes and bubbles. Let me at this point quote some, especially for the Netherlands relevant 'history is repeating itself' anecdotes.

The first concerns the well-known history srrounding the inflation and consequently steep drop in the price of tulip bulbs in 17th Century Holland. At the peak of Tulipomania (Amsterdam, 1636–1637) 1 bulb of viceroy was sold for:

Two lasts of wheat	448
Four lasts of rye	558
Four fat oxen	480
Eight fat swine	240
Twelve fat sheep	120
Two hogsheads of wine	70
Four tuns of beer	32
Two tuns of butter	192
One thousand lbs. of cheese	120
A complete bed	100
A suit of clothes	80
A silver drinking cup	60

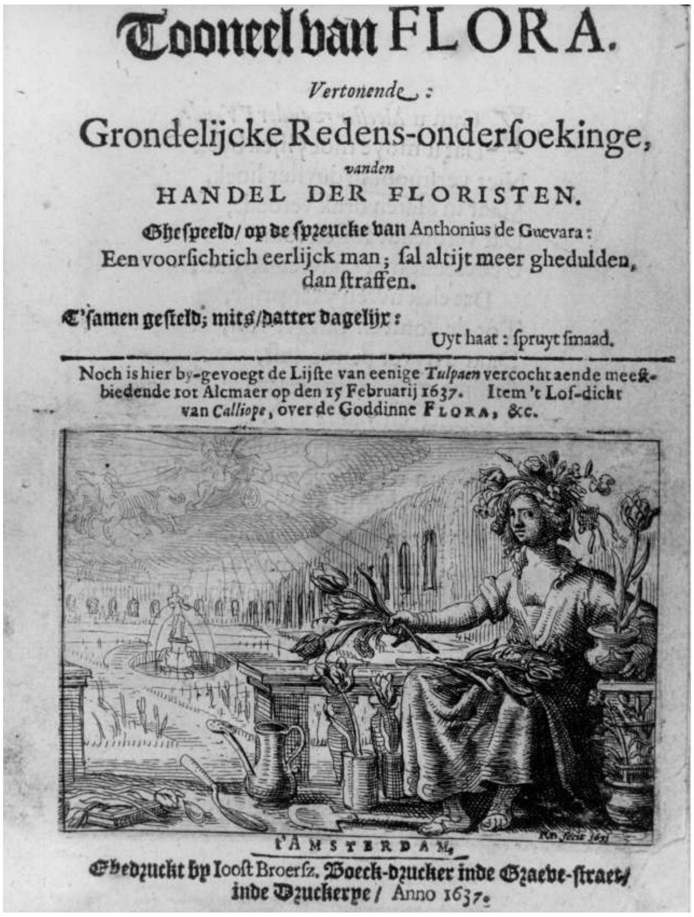
amounting to the considerable total of 2500 florins. The reason that such prices were paid lay in the fact that homogeneous coloured tulips at some (random?) time in the future could 'break' and transform into a highly non-homogeneous multi-coloured very rare species. As such, the buyer was investing into the future random payout and prepared to pay handsomly for that. Just before the bubble burst, tulip bulbs were sold forward (like futures today). It may be disputed to what level irrationality was present in the tulip market of 17th Century Amsterdam. One of the ways prices were determined involved assistants of the buyer and the seller to negotiate the price standing in an inn within a small circle drawn on the ground. This circle was referred to as 'het ootje'. Hence linguistic history seems to have chosen for considerable irrationality if we are go to by todays interpretation of 'in het ootje nemen'. It was only realised on the 20th century that the magical breaking of tulips was due to a virus. A most readable account of the history of the tulip from an early day Turkish delight to the present day Dutch emblem is to be found in Pavord [24].

A second Dutch historical account showing that serious option trading has been with us for centuries is to be found in Joseph de la Vega's *Confusión de Confusiones* [13]. Joseph de la Vega was a 17th Century businessman living in an Amsterdam community of Portugese Jews having fled from the Spanish Inquisition. He recounts the following discussion on the floor of the Amsterdam stockmarket ('beurs') by the end of the 17th Century:

"If I may explain 'opsies' [further, I would say that] through the payment of the premiums, one hands over values in order to safeguard one's stock or to obtain a profit. One uses them as sails for a happy voyage during a beneficent conjucture and as an anchor of security in a storm.

The price of the shares is now 580, [and let us assume that] it seems to me that they will climb to a much higher price because of the extensive cargoes that are expected from India, because of the good business of the Company, of the reputation of its goods, of the prospective dividends, and of the peace in Europe. Neverthesee I decide not to buy shares through fear that I might encounter a loss and might meet with embarrassment if my calculations should prove erroneous. I therefore turn to those persons who are willing to take options and ask them how much [premium] they demand for the obligation to deliver shares at 600 each at a certain later date. I come to an agreement about the premium, have it transferred [to the taker of the options] immediately at the Bank, and then I am sure that it is impossible to lose more than the price of the premium. And I shall gain the entire amount by which the price [of the stock] shall surpass the figure of 600.

In case of a decline, however, I need not be afraid and disturbed about my honor nor suffer fright which could upset my equanimity. If the price of the shares hangs around 600, I [may well] change my mind and realize that the prospects are not as favorable as I had presumed. [Now I can do one of two things.] Without danger I [can] sell shares [against time], and then every amount by which they fall means a profit. [Or I can enter into



A pamphlet that warns against the speculative trade in tulips. C. van der Woude: *Tooneel van flora.*, 1637. Copyright: Bibliotheek Wageningen UR, Speciale Collecties another option contract. In the earlier case] the receiver of the premium was obliged to deliver the stock at an agreed price, and with a rise in the price I could lose only the bonus, so now I can do the same business (in reverse), if I reckon upon a decline in the price of the stock. I now pay premiums for the right to deliver stock at a given price.”

Hence the above quote contains the notions of put and call together with the risk management consequences of buying or selling such products. De la Vega further discusses the notion of shortselling.

I would like to add that the edition [13] also contains the most interesting *Extraordinary Popular Delusions and the Madness of Crowds* by Charles MacKay, written in 1841. His text clearly shows that ‘there is nothing new under the sun’ when it comes to bubbles and crashes, greed, irrationality, herding and market psychology. Every student interested in financial markets ought to read these historical accounts. A final comment I would like to make however. In all analyses of bubbles and crashes one has to be careful in too quickly filing such events par default in the chapter on irrational behaviour. A much more detailed study on the specific case at hand is always warranted. This also holds true for the Tulip Bubble. Garber [14] offers a market-fundamental explanation of the latter, as well as for two other bubbles also discussed by MacKay [13], the Mississippi Bubble (1719–1720) and the closely connected South Sea Bubble (1720).

Some thoughts on the present and the future

By now, the mathematical theory of financial markets is highly developed and well understood. Without wanting to make a link to econophysics, many compare the present state of the theory with the power of Newtonian mechanics used for describing nature in a first approximation. I personally think that we are not there yet; too many really fundamental practical issues remain too little understood. We may understand markets in a ‘normal’ state, however we have little to go by with that same theory when the very important ‘abnormal/extreme’ situations occur. The whole field of international market regulation, through globally accepted principles for quantitative risk management, is precisely interested in these ‘bad case scenarios’. For a brief introduction in some of the issues mathematicians ought to be aware about, see Embrechts [11]. Also the development of new markets puts a challenge on the mathematical theory now available. I am for instance thinking of derivatives in insurance markets (see for instance Lane [19]), the deregulation of energy markets and real option markets (Davis et al. [8]) to name some of the more important ones. In all of these, besides the modelling of a price process, one also has to model an underlying physical process with all the added intricacies; as a prime example in the case of energy derivatives, think of the modelling of electricity transportation and storage. Because there is no effective way to store electricity, one cannot construct arbitrage portfolios with the underlying commodity and hence one needs to model the so-called term structure of future prices directly. Also, supply and demand fundamentals translate directly into spot price behaviour leading to a mean reverting spot process with spikes. These markets also lead in a very natural way to highly complex contract structures with implicit options. An example are the so-called swing options which are American style and have a path-dependent pay-off structure. A swing option gives the holder of the options the right to buy power on a daily basis during the lifetime of the contract (30 days, say). There is an upper limit for the number of days at which exercise is allowed (20 days, say). The strike price may be fixed (F, say) or floating and typically a distinguishing feature, like a volumetric constraint, is present. Examples of such constraints are:

- a maximum flow rate, R_t;
- a monthly minimum demand m, maximum demand M and a daily maximum demand D, all quoted as percentages of the theoretically possible energy consumption over the respective time intervals ($M \leq 20$), and
- a ‘take or pay’ constraint: the failure to take m by the end of the month costs $(m - C)F_m$ where F_m is an agreed unit price,
$$C = \int_0^{30} V_s ds,$$
where V_t denotes the instantaneous consumption rate at time t.

Using these constraints, the swing option can now be defined precisely (mathematically). Let

$$T = \{= (\tau_1, \ldots, \tau_{20}) \mid \tau_i \text{ stopping times}, \tau_1 < \cdots < \tau_{20}\}$$

Moreover,

$$K_\tau = \Bigg\{ V : [0,30] \times \Omega \to [0, R_t] \mid V \text{adapted}, \\ V \equiv 0 \text{ on } \left(\bigcup_{i=1}^{20} [\tau_i, \tau_i + 1) \right)^c \cap [0,30], \\ C \leq M, \int_{\tau_i}^{\tau_i+1} V_s ds \leq D, \quad i = 1, \ldots, 20 \Bigg\}.$$

With this notation, the value of the swing contract can be described as

$$\sup_{\tau \in T} \sup_{V \in K_\tau} E^Q \left[\int_0^{30} (S_t - F)_+ V_t e^{-rt} dt - (m - C)_+ F_m e^{-30r} \right].$$

The latter formula, and indeed the underlying spot market, are a far cry from their originators (1), (11) and (10). The mathematical theory however needed for the pricing of swing options is available. What is much less understood are for instance the properties of the spot market (S_t) and the appropriate choice of Q, to name just two.

These, and many more derivative products will be engineered further in the future. Besides the intrinsic modelling of the underlying markets, at the end of the day risks taken will have to be aggregated and managed. This is where integrated risk management enters; see Crouhy et al. [7] for an excellent account. In a banking context, at the close of trading each day, a so-called P&L (Profit-and-Loss) is determined and projected (estimated) as an unknown distribution function F, a specific number of days (typically 10) in the future. Based on this F, risk measures are calculated, like the famous VaR (Value-at-Risk) which, for a given level $0 < \alpha < 1$, is ‘just’ the quantile:

$$\mathrm{VaR}(\alpha) = -F^{-1}(\alpha) \tag{16}$$

where α is typically small (corresponding to losses), $\alpha = 0.05$, 0.01. Mathematicians have contributed in a fundamental way to questioning and understanding the rationale behind the choice of (16) (see for instance Artzner et al. [2]). For Dutch scientists, the discussion around using (16) as a risk measure is a déjà-vu. Indeed, following the dyke disaster of February 2, 1953, the Delta project demanded as a safety margin (α) for the dyke heights a 1/4000-year level for the delta region and the north and a 1/10000-year level for the 'Randstad'. Recall that the storm causing the 1953 flood was a 1/300-year event! Also in this case, the estimation of risk measures (dyke heights) given an α level ($1/t$-year) is one issue, equally important and much harder to settle is the choice of α, respectively t.

I very much hope that my paper will contribute in making mathematicians also interested in the latter problem. Financial derivatives are here to stay. They form an integral part of our social welfare society/system and hence should be understood and risk managed in a scientifically sound way. If mathematicians can really contribute to the global understanding of modern financial markets, then these Wizards of Wall Street will no doubt have an impact. To what extend mathematics has changed (or is changing) finance will to a large extend depend on how deeply mathematicians are prepared to get involved with the wider issues.

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