Problemen

| Problem Section

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

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Problem 30 (H.J.A. Duparc)
In general, if you interchange the hands of a clock you don't get a valid time. How many times a day can you interchange the hands of a clock and still get a valid time?

Problem 31 (Sander Dahmen)
Define for each positive integer n the regular $(n+1)$-dimensional solid $R_n = \{x \in \mathbf{R}^{n+1}\colon \sum_{i=1}^{n+1} |x_i| \leq 2\}$ and the n-dimensional hyperplane $V_n = \{x \in \mathbf{R}^{n+1}\colon \sum_{i=1}^{n+1} x_i = 0\}$. Compute the n-dimensional volume of $R_n \cap V_n$.

The following three problems are proposed by Wlodek Kuperberg.

Problem 32
Show that you can cover a square by three smaller squares, but not by two smaller squares.

Problem 33
Show that you can cover an n-dimensional cube by $n+1$ smaller cubes.

Problem 34 (Open problem)
Can you cover an n-dimensional cube by n smaller cubes?

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Problem 24
Suppose that T is a triangular billiard table with sharp angles. Show that you can put a billiard ball on the table and shoot it in such a way that its (infinite) trajectory is a triangle.

Solutions by Frits Beukers, L. Bleijenga, Ruud Jeurissen, Floor van Lamoen, Jack van Lint. Most contributors mention that the solution is the orthic triangle. The inside out solution below is given by Frits Beukers.
Let T be the triangle corresponding to the billiard table. Let S be the triangle which is the orbit of the ball. Then the sides of T consist of the exterior angle bisectors of S. Suppose that the vertex angles of S are α, β, γ. Then an easy computation shows that the vertex angles of T are $(\pi-\alpha)/2, (\pi-\beta)/2, (\pi-\gamma)/2$. These are all sharp angles, which can be chosen arbitrarily by a suitable choice of α, β, γ. Hence any sharp angled triangle T occurs.

Problem 25 (Ruud Jeurissen)
Let V and W be finite disjoint sets of cardinality m and n, respectively, with $m \leq n$. Let Q be a set of quadruples each containing 2 elements from V and 2 elements from W, such that no two of its quadruples have 3 elements in common.
a) Prove that $|Q| \leq \binom{m}{2}\lfloor\frac{n}{2}\rfloor$.
b) Prove that the bound in a) is sharp.

Solutions by Sybren Botma, Ruud Jeurissen and Jack van Lint. Two pairs in W that form a quadruple with the same pair in V must be disjoint. So each of the $\binom{m}{2}$ pairs in V can be combined with at most $\lfloor\frac{n}{2}\rfloor$ pairs in W, which proves (a). The problem is to show

that the bound is sharp. Jeurissen's solution uses Latin squares. Botma's solution is by a direct construction. Below is Van Lint's solution, which uses Baranyai's theorem.
Let k be even. By Baranyai's theorem (cf. Van Lint and Wilson, *A Course in Combinatorics*, Theorem 38.1) the set of $\binom{k}{2}$ pairs from a k-set can be split into $k-1$ disjoint parallel classes. Here, a parallel class is a set of $k/2$ pairs which form a partition of the set. First assume that both m and n are even. Apply Baranyai's theorem to V and W to obtain $m-1$, respectively $n-1$ parallel classes of pairs. Since $m \leq n$, there is a one-to-one mapping from the set of parallel classes from V to a subset of the set of parallel classes from W. Now, let Q consist of all fourtuples $\{a,b,c,d\}$ where $\{a,b\}$ runs through all pairs from V and for a fixed pair $\{a,b\}$, the pair $\{c,d\}$ runs through the pairs in the parallel class from W corresponding to the parallel class from V to which $\{a,b\}$ belongs. We find $\binom{m}{2} \cdot \frac{n}{2}$ fourtuples which clearly satisfy the conditions for the set Q. If n is odd, add an extra element ∞ to W to make it even and the argument runs through. So we may assume that n is even. If m is odd, pick an element of W and add a copy of it to V and again the argument runs through.

Problem 26 (F. Roos)
Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solution This problem has been solved by Frits Beukers, Jaap Spies and Christiaan van de Woestijne. The solution shall appear in the next issue of Nieuw Archief.