Frits Beukers
Mathematisch Instituut, Universiteit Utrecht
Postbus 80010, 3508 TA Utrecht
beukers@math.uu.nl

Weia Reinboud
Simon Bolivarstraat 87, 3573 ZK Utrecht
weia@antenna.nl

Snellius versneld

In 1621 ontdekte Snellius een eenvoudige manier om de benadering van π van Archimedes aanmerkelijk te verbeteren. Dit bleek een diepzinnig resultaat dat pas later werd bewezen. Frits Beukers en Weia Reinboud, respectievelijk docent en studente aan de Universiteit Utrecht, plaatsen de methode in een breder verband en ontwikkelen andere, op Snellius' methode geïnspireerde formules.

Practically all computations of the value of π before 1600 were done using Archimedes' method. This method consists of approximation of the circle with diameter 1 by inscribed and circumscribed regular polygons. Denote the circumference of the inscribed and circumscribed regular N-gon by P_N and Q_N respectively. Then $P_N < \pi < Q_N$ and

$$\lim_{N\to\infty} P_N = \lim_{N\to\infty} Q_N = \pi.$$

A little trigonometry shows us that

$$Q_N = N\tan\frac{\pi}{N}, \quad P_N = N\sin\frac{\pi}{N}, \tag{1}$$

from which the duplication formulae

$$Q_{2N} = \frac{2P_NQ_N}{P_N+Q_N}, \quad P_{2N} = \sqrt{P_NQ_{2N}} \tag{2}$$

follow readily. For more details, see [3].

Archimedes started with the values $Q_6 = 2\sqrt{3}$ and $P_6 = 3$ and calculated $Q_{12}, P_{12}, Q_{24}, \ldots, Q_{96}, P_{96}$ consecutively using the duplication formulae in (2). See also [1].

Ludolph van Ceulen, around 1600, continued this procedure until he obtained 35 decimal places of π. To get an idea of the accuracy of the approximation Q_N to π we use the Taylor series expansion of $\tan x$. We get,

$$\begin{aligned} Q_N &= N\left(\frac{\pi}{N} + \frac{1}{3}\left(\frac{\pi}{N}\right)^3 + \frac{2}{15}\left(\frac{\pi}{N}\right)^5 + \cdots\right) \\ &= \pi + \frac{1}{3}\frac{\pi^3}{N^2} + \frac{2}{15}\frac{\pi^5}{N^4} + \cdots \end{aligned}$$

In other words, $Q_N - \pi$ has order of magnitude $O(\frac{1}{N^2})$. More precisely, $Q_N - \pi$ is equal to $\frac{\pi^3}{3N^2}$ up to order $O(\frac{1}{N^4})$. Similarly, $P_N - \pi$ equals $-\frac{\pi^3}{6N^2}$ up to order $O(\frac{1}{N^4})$. From these two facts it follows immediately that

$$\frac{1}{3}(Q_N - \pi) + \frac{2}{3}(P_N - \pi) = \frac{1}{3}Q_N + \frac{2}{3}P_N - \pi$$

has order $O(\frac{1}{N^4})$. So we see that $\frac{1}{3}Q_N + \frac{2}{3}P_N$ gives an approximation of π having approximately twice as many correct digits as P_N or Q_N. This was discovered by the Dutch natural scientist Willebrord Snellius in 1621, about ten years after Van Ceulen's death. Snellius used geometrical observations to find his approximations. Only much later Christiaan Huygens delivered a complete proof of the correctness of these observations.

The conclusion is that Van Ceulen could have stopped halfway through his calculations, compute $\frac{1}{3}Q_N + \frac{2}{3}P_N$ for the value of N then reached, and obtain 35 decimal places of π. Such a speedup of calculation makes one wonder if Snellius' discovery can be generalised in its turn. It is the purpose of this article to give a number of such generalisations.

In our considerations we assume that we carry out a number of steps of the Archimedean algorithm. We then stop and collect the values of P_N, Q_N. Now use one of the theorems in this article. For example, after we have done the Archimedean steps, we find, up to 39 decimal places,

$$\begin{aligned} P_{96} &= 3.141031950890509638111352926459660107036 \\ Q_{96} &= 3.142714599645368298168859093772123871001 \\ \pi &= 3.141592653589793238462643383279502884197 \end{aligned}$$

We see that Archimedes approximation is correct up to two decimal places. Let us now use Archimedes result and, for example, theorem 1, which states that

$$\pi = Q_N - \frac{Q_N}{3}\left(\frac{Q_N}{N}\right)^2 + \frac{Q_N}{5}\left(\frac{Q_N}{N}\right)^4 - \frac{Q_N}{7}\left(\frac{Q_N}{N}\right)^6 + \cdots$$

Afbeelding rechts. Willebrord Snellius (1580–1626). Ets, 1600–1625, collectie Museum Boerhave.


WILLEBRORDUS SNELLIUS.
PROFESSOR MATHESEOS.

This series is easily obtained from the power series expansion of arctan. Using $N = 96$ and our value of Q_{96} summation of 12 terms of this series yields the approximation

$$3.14159265358979323846264338327950288390 9$$

which is correct up to 35 decimal places. So Archimedes' four steps and addition of 12 terms in a series would have sufficed to get Van Ceulen's precision.

More generally, suppose we wish to compute π to L decimal places. As a time unit we may take the time to perform one operation (addition, multiplication, division) of two L-digit numbers. Then the Archimedean algorithms gives us the answer with L-digit precision in $O(L)$ steps. However, in the last section of this article we see that if we combine the Archimedean steps with a Snellius' type acceleration, we require only $O(\sqrt{L})$ steps.

We have not made any effort to make this very precise, since modern day methods are far better suited for the high precision calculation of π. The most widely used are the Gauss-Salamin procedure or the Ramanujan-Chudnovsky series, which are very easy to program. Again, see [4] or [3].

The motivation for this note is not to present a speedup for the calculation of π. Instead, we thought it remarkable that generalisations of Snellius' trick could be written down as a handsome series, with terms that look quite simple. In the text one finds series expressions for $\frac{\pi}{Q_N}$ in terms of $\frac{Q_N}{N}$ and $\frac{P_N}{N}$ in theorems 1 and 6. Of course there also exists a power series expansion for $\frac{\pi}{P_N}$ in terms of $\frac{Q_N}{N}$ and $\frac{P_N}{N}$. However, the first one will be a very ugly one, which we haven't even attempted to write down, whereas the second looks very simple and is stated in theorem 4.

Of course the possiblity of improvements, like the ones discussed in this paper, has been considered by many others, professional mathematicians and amateurs alike. See, for example [5].

Unfortunately it is hard to get a good overview concerning publications on this subject. So we do not claim any originality in the results here. We present it as a hopefully amusing aside to π-folklore.

Accelerations based on arctan

The first improvement is obtained by using the arctangent series. From $Q_N = N \tan \frac{\pi}{N}$ it follows immediately that

$$\frac{\pi}{Q_N} = \frac{\arctan(Q_N/N)}{Q_N/N}.$$

Using the well-known Taylor series for arctan we obtain our first result.

Theorem 1. *We have*

$$\frac{\pi}{Q_N} = 1 - \frac{1}{3}\left(\frac{Q_N}{N}\right)^2 + \frac{1}{5}\left(\frac{Q_N}{N}\right)^4 - \frac{1}{7}\left(\frac{Q_N}{N}\right)^6 + \cdots$$

So, by subtracting $Q_N \cdot \frac{1}{3}\left(\frac{Q_N}{N}\right)^2$ from Q_N Van Ceulen could have doubled the precision of his calculations in one stroke. By adding the next term the precision could have been tripled. There is a nice variation on the above formula which does not use P_N/N or Q_N/N, but only the value $t_N = \frac{Q_N - P_N}{2Q_N}$. To explain this we need a few facts on hypergeometric functions.

Let a, b, c be real numbers and $c \neq 0, -1, -2, \ldots$ Then the Gauss' hypergeometric function with parameters a, b, c is defined by the power series

$$F_{\text{Gauss}}(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.$$

Here, $(x)_k$ is the so-called Pochhammer symbol defined by $(x)_k = x(x+1)\cdots(x+k-1)$. The series converges for all complex z with $|z| < 1$. Here is one of the many transformation formulas between hypergeometric series:

$$F_{\text{Gauss}}\left(\frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}, \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}; \frac{4t^2 - 4t}{(1-2t)^2}\right) = (1-2t)^a F_{\text{Gauss}}\left(a, b, \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}; t\right). \quad (3)$$

This formula is basically due to Kummer and can be found in [2, page 561]. As a result of this formula we find the following application which is useful for us.

Proposition 2. *We have*

$$F_{\text{Gauss}}\left(\frac{1}{2}, 1, \frac{3}{2}; \frac{4t^2 - 4t}{(1-2t)^2}\right) = 1 - \sum_{k=1}^{\infty} \frac{(2)_k}{(3/2)_k} \frac{t^k}{k}.$$

Proof. To see this, apply formula (3) with $a = b = 1$ to get

$$F_{\text{Gauss}}\left(\frac{1}{2}, 1, \frac{3}{2}; \frac{4t^2 - 4t}{(1-2t)^2}\right) = (1-2t) F_{\text{Gauss}}\left(1, 1, \frac{3}{2}; t\right) = (1-2t)\sum_{k=0}^{\infty} \frac{k!}{(3/2)_k} t^k.$$

For any $k \geq 1$ the coefficient of t^k in the last product is of course equal to

$$\frac{k!}{(3/2)_k} - 2\frac{(k-1)!}{(3/2)_{k-1}}.$$

A straightforward calculation shows that this is equal to

$$-\frac{(k+1)!}{(3/2)_k}\frac{1}{k} = -\frac{(2)_k}{(3/2)_k}\frac{1}{k}.$$

Our proposition now follows immediately. □

We note that the arctangent series is an example of a hypergeometric series. One easily checks that

$$\frac{\arctan z}{z} = F_{\text{Gauss}}\left(\frac{1}{2}, 1, \frac{3}{2}; -z^2\right).$$

We wish to substitute $z = Q_N/N$. Now observe that $t_N = (Q_N - P_N)/2Q_N = \frac{1}{2}(1 - \cos\frac{\pi}{N})$. Using this, we find that

$$-\left(\frac{Q_N}{N}\right)^2 = -\left(\frac{\sin\frac{\pi}{N}}{\cos\frac{\pi}{N}}\right)^2 = 1 - \frac{1}{(\cos\frac{\pi}{N})^2} = 1 - \frac{1}{(1-2t_N)^2} = \frac{4t_N^2 - 4t_N}{(1-2t_N)^2}.$$

Using this observation and proposition 2 we find another result.

Theorem 3. *We have*

$$\frac{\pi}{Q_N} = 1 - \sum_{k=1}^{\infty} \frac{(2)_k}{(3/2)_k} \frac{t_N^k}{k}.$$

Note that if we take the first two terms of this series, we get

$$\pi \approx Q_N - \frac{4}{3}Q_N t_N = Q_N - \frac{2}{3}(Q_N - P_N) = \frac{1}{3}Q_N + \frac{2}{3}P_N,$$

which is precisely Snellius' improvement. Finally we like to point out that $t_N = 2(P_{2N}/2N)^2$, so we see that t_N is of order $O(\frac{1}{N^2})$.

Accelerations based on arcsin

Just as with the arctan series we can also play with the arcsin series, which reads

$$\begin{aligned}\frac{\arcsin z}{z} &= \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!}\frac{z^{2k}}{2k+1}\\ &= F_{\text{Gauss}}\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};z^2\right).\end{aligned}$$

As an immediate consequence, the next theorem follows from $\frac{\pi}{P_N} = \frac{\arcsin(P_N/N)}{P_N/N}$.

Theorem 4. *We have*

$$\frac{\pi}{P_N} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!}\frac{1}{2k+1}\left(\frac{P_N}{N}\right)^{2k}.$$

There is a small variation based on the following proposition.

Proposition 5. *We have*

$$(1-z^2)^{1/2}\frac{\arcsin z}{z} = 1 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(k-1)!}{(3/2)_k}z^{2k}.$$

Proof. This result follows from the another well-known formula in hypergeometric functions, which reads

$$(1-z)^{a+b-c}F_{\text{Gauss}}(a,b,c;z) = F_{\text{Gauss}}(c-a,c-b,c;z).$$

The formula can be found in [2, page 559]. Apply this with $a = b = 1/2, c = 3/2$ and z replaced by z^2 to get

$$(1-z^2)^{-1/2}\frac{\arcsin z}{z} = F_{\text{Gauss}}(1,1,\frac{3}{2};z^2).$$

Multiply on both sides by $1-z^2$ and notice that

$$\begin{aligned}(1-z^2)^{\frac{1}{2}}F_{\text{Gauss}}(1,1,\frac{3}{2};z^2) &= (1-z^2)\sum_{k=0}^{\infty}\frac{k!}{(3/2)_k}z^{2k}\\ &= 1 + \sum_{k=1}^{\infty}\left(\frac{k!}{(3/2)_k} - \frac{(k-1)!}{(3/2)_{k-1}}\right)z^{2k}\\ &= 1 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(k-1)!}{(3/2)_k}z^{2k}. \qquad \square\end{aligned}$$

We apply our proposition with $z = P_N/N = \sin\frac{\pi}{N}$. Notice that $(1-z^2)^{1/2} = \cos\frac{\pi}{N}$. Hence the proposition implies the following.

Theorem 6. *We have*

$$\frac{\pi}{Q_N} = 1 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(k-1)!}{(3/2)_k}\left(\frac{P_N}{N}\right)^{2k}.$$

Looking back we see that we expressed $\frac{\pi}{Q_N}$ as a power series in $\frac{P_N}{N}$ and in $\frac{Q_N}{N}$. We also expressed $\frac{\pi}{P_N}$ as a power series in $\frac{P_N}{N}$. Although there is certainly a power series for $\frac{\pi}{P_N}$ in terms of $\frac{Q_N}{N}$, the shape of this series does not seem to be as simply as the other three.

Gain of the acceleration

In this section we indicate briefly how much Archimedes' calculation can be speeded up using our Taylor series. As we said before, we use the time taken for one operation on two L-digit numbers as a unit of time. Suppose we wish to calculate π to L decimal places. We first carry out $\sqrt{L}$ steps of Archimedes' algorithm. This gives us $\sqrt{L}\cdot\log_{10}(4)$ correct decimal places. To increase this precision by a factor $\sqrt{L}$ we have to take $O(\sqrt{L})$ terms of any of the power series given in the previous sections. So the total number of steps is again $O(\sqrt{L})$.

As we already indicated in the introduction, modern methods like the Gauss-Salamin algorithm or its speedup by the Borweins provide a much faster scheme of computation. The number of steps required for the latter methods is $O(\log L)$.

References

1 Archimedes, Measurement of the circle, reprinted in [4].

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4 L. Berggren, J. Borwein, P. Borwein, *Pi: a source book*, Springer Verlag 1997.

5 G.M. Philips, Archimedes, the numerical analyst, American Math. Monthly 88 (1981), 165–169. Reprinted in [4].