Problemen

| Problem Section

Problem 24

Suppose that T is a triangular billiard table with sharp angles. Show that you can put a billiard ball on the table and shoot it in such a way that its (infinite) trajectory is a triangle.

Problem 25 (Ruud Jeurissen)

Let V and W be finite disjoint sets of cardinality m and n, respectively, with $m \leq n$. Let Q be a set of quadruples each containing 2 elements from V and 2 elements from W, such that no two of its quadruples have 3 elements in common.

a) Prove that $|Q| \leq \binom{m}{2}\lfloor\frac{n}{2}\rfloor$. Here $\binom{m}{2}$ denotes a binomial and $\lfloor\frac{n}{2}\rfloor$ means the greatest integer $\leq \lfloor\frac{n}{2}\rfloor$.

b) Prove that the bound in a) is sharp.

Problem 26

Does there exist a triangle with sides of integral lengths such that its area is equal to the square of the length of one of its sides?

Solutions to volume 2, number 2 (June 2001)

Problem 18 (Alex Heinis)

Let a, k be positive numbers. Count the number of maps $f\colon \mathbf{N} \to \mathbf{N}$ such that $f^k(n) = n + a$ for all $n \in \mathbf{N}$.

Solution Ruud Jeurissen (Nijmegen) and Lute Kamstra (Amsterdam) give similar solutions. If k divides a, the number of maps is $a!/\frac{a}{k}!$. If k does not divide a, there are no such maps.

Let $f^k(n) = n + a$ for all $n \in \mathbf{N}$. Then $f(n+a) = f(f^k(n)) = f^k(f(n)) = f(n) + a$, so f is determined by its values on $A := \{1, 2, \ldots, a\}$.
Define $F : A \to A$ by $F(m) = f(m) \bmod a$. Using $f(n-a) = f(n)$ for $n > a$ we see that $F^k(m) = m$ for all $m \in A$. Let $c \in A$ and consider the sequence $c = c_1, c_2, c_3, \ldots$ with $F(c_i) = c_{i+1}$; it is periodic and its smallest period d is a divisor of k. There are non-negative integers m_i with $f(c_i) = c_{i+1} + m_i a$, and so $f^2(c) = f(c_2 + m_1 a) = f(c_2) + m_1 a = c_3 + m_1 a + m_2 a$. Continuing we find $c + a = f^k(c) = c + (m_1 + m_2 + \cdots + m_k)a$. It follows that $m_i = 0$ except for one i for which it is 1, and from this that the period is k. Apparently the powers of F form a cyclic group acting on A, all orbits have cardinality k and in every orbit there is one element x with $f(x) = F(x) + a$, whereas for the other elements y we have $f(y) = F(y)$.
We now know that if k does not divide a, the answer is 0.
Let $k|a$. Let $\{c_1, c_2, \ldots, c_k\}$ be an orbit of F, the numbering being chosen such that $f(c_i) = c_{i+1}$ for $i \neq k$ and $f(c_k) = c_1 + a$. Thus the elements of A form $\frac{a}{k}$ sequences of length k. This can happen in $a!/\frac{a}{k}!$ ways.
Conversely, let S be such a set of $\frac{a}{k}$ sequences. Define the function $f : \mathbf{N} \to \mathbf{N}$ as follows: for each $(c_1, c_2, \ldots, c_k) \in S$ let $f(c_i) = c_{i+1}$ for $i < k$ and $f(c_k) = c_1 + a$; for $n > a$ with $n = c + ma$, $c \in A$, let $f(n) = f(c) + ma$. One easily checks that indeed $f^k(n) = n + a$ for all $n \in \mathbf{N}$.
So if $k|a$, the answer is $a!/\frac{a}{k}!$.

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

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Problem 19

A billiard table T has the shape of a regular pentagon, with sides of length of 1 meter. You can put a billiard ball anywhere on the table and shoot it in any direction over a distance of 10 meters. Give the maximal number of times the ball can hit a side.

This problem remains open. Most probably the trajectory that hits the pentagon at consecutive mid points gives the maximal number of reflections. Billiard problems may seem easy, but in fact they are very hard. Problem 24 above, for instance, is open for triangular billiard tables with an obtuse angle.

Problem 20 (A.F. Tiggelaar)

Suppose that ABC and DEF are two triangles in $\mathbf{R}^3$ and that V is the plane that contains ABC. Suppose that AD, BE and CF are perpendicular to V and that $AD = BC$, $BE = AC$, $CF = AB$. Construct the point in V that is equidistant to D, E and F.

Solution By Ruud Jeurissen (Nijmegen), L. Bleijenga (Den Haag), Jos Brakenhoff (Leiden), M. Hendriks (Elst), Hans Linders (Eindhoven). Brakenhoff's solution is below.

The subset $S_{D,E} \subset V$ of points that are equidistant to D and E is a line. We have that

$$CD^2 = AD^2 + AC^2 = BC^2 + BE^2 = CE^2 \text{ so } C \in S_{D,E}.$$

Let C' be the projection of C onto AB. Then

$$C'D^2 = CD^2 - C'C^2 = CE^2 - C'C^2 = C'E^2 \text{ so } C' \in S_{D,E}.$$

Since $S_{D,E}$ is a line, $S_{D,E}$ has to be the perpendicular from C. By symmetry, $S_{D,F}$ and $S_{E,F}$ are the perpendiculars from B and A, respectively. The point that is equidistant to D, E and F is the orthocenter of ABC.