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Letters to the editor

Oscillations of the Taylor polynomials of the sine function

In *Nieuw Archief voor Wiskunde*, december 2000, F. Rothe gives a first-order estimation of the number of zeros N_ω of a Taylorpolynomial P_ω of the sine function of high order ω, i.e. $N_\omega \backsim \frac{2}{\pi e}\cdot\omega$. The asymptotic formula is proved by joining to the common upper bound of the remainder a sufficiently close lower bound. The bound and its derivation are clearly ad-hoc.

It seems to me that the result is a straightforward corollary to two general formulae for remainders of Taylor expansions I derived in [1, Proposition 3.2, Theorem 3.5] and [2, Theorem 3.11 and Chapter 3.4]. The formulae belong to nonstandard analysis, which makes them somewhat particular. However, the formulae contain two parameters, index and place. It is not natural to let one of the parameters go to infinity, or to couple them, so the formulae cannot be translated directly to standard formulae.

The formulae were derived within Internal Set Theory (IST) of E. Nelson. In this axiomatic system, which is consistent with classical set theory, the set **R** contains infinitesimals and infinitely large or unlimited numbers. Below I use some symbols of nonstandard asymptotics: $\varnothing$, the 'unknown' infinitesimal, £ the 'unknown' limited number, @ the 'unknown' appreciable number, analogous to respectively $o(1)$, $O(1)$ and $O_s(1)$. An up-to-date introduction to IST is given in [3].

In the case of the sine function, the first formula is the following. Let $R_{\omega-2}(x)$ be the remainder

$$\sin x - \sum_{n=0}^{(\omega-3)/2} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

such that the first neglected term $(-1)^\omega \frac{x^\omega}{\omega!}$ is of (odd) degree ω. Then, for all real x, limited or unlimited

$$R_{\omega-2}(x) = \frac{(1+\varnothing)}{1+(x/\omega)^2}\cdot(-1)^\omega\frac{x^\omega}{\omega!}. \qquad (1)$$

The second formula gives the remainder for the special values $x=\omega'+u$ with $\omega'=\frac{\omega}{e}+\frac{1}{2e}\log\omega$ and u limited. Then

$$R_{\omega-2}(\omega'+u)\simeq(-1)^\omega\frac{e^{eu}}{\sqrt{2\pi}(1+\frac{1}{e^2})}. \qquad (2)$$

Applied to the cosine function, the formulae are identical, for even ω.

We consider only positive x and claim that the distance between the last zero of the Taylor polynomial $P_{\omega-2}$ and ω' is limited. Notice that, firstly,

$$F(x)\equiv\frac{1}{1+(x/\omega)^2}\cdot(-1)^\omega\frac{x^\omega}{\omega!}$$

is monotonous in x, and, secondly that formula (2) implies that if u passes through all limited values, $F(\omega'+u)$ passes through all appreciable values. Hence it follows from the monotony of $F(x)$ that $F(x)$ is infinitesimal for $x=\omega'+u$ with u negative unlimited, appreciable for $x=\omega'+u$ with u limited, and unlimited for $x=\omega'+u$ with u positive unlimited. The same partition holds for the remainder, since $R_{\omega-2}(x)=(1+\varnothing)F(x)$.

As long as the remainder of the sine function is infinitesimal, the Taylor polynomial has a zero infinitely close to every zero of the sine function $k\pi$, with integer k. In fact, there is only one zero, for the polynomial $P_{\omega-2}$ is transversal to the x-axis: the derivative of the polynomial is the Taylor polynomial of degree $\omega-3$ of the cosine function, and because formula (1) is also valid for the cosine function, one has $P'_{\omega-2}(x)\simeq\cos(x)\simeq\pm1$ for all $x\simeq k\pi$. If the remainder of the sine function is unlimited, the distance between $P_{\omega-2}$ and the x-axis is also unlimited, and no more zeros can occur. We conclude that the last zero is of the form $\omega'+u$ with u limited. It could be determined individually for most ω up to an infinitesimal, using formula (2).

We see that the number of positive zeros is $(\omega'+£)/\pi$, and by symmetry

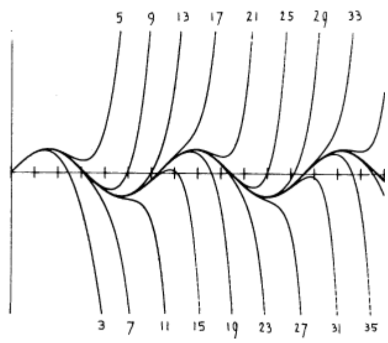
$$N_{\omega-2}=\frac{2}{\pi e}\cdot\omega+\frac{1}{\pi e}\cdot\log\omega+£.$$

The same expression holds for N_ω.

The problem in question is somewhat anecdotical, but the formulae behind are not. Formulae (1) and (2) hold under very general conditions, implying that the remainder as a function of the distance to the origine is locally exponential such as in (2), and that the series is locally exponential (geometric) as a function of the index; indeed, we recognize in (1) the remainder of a geometric series with ratio $-\left(\frac{x}{\omega}\right)^2$. Finally we mention that that the crucial values $x=\omega'+£$ may be determined from (1), by solving the *external equation*

$$\frac{(1+\varnothing)}{1+(x/\omega)^2}\cdot\frac{x^\omega}{\omega!}=@,$$

which can be effectuated using systematic, algebraic methods based on the *external numbers* of [3] and [4].



The sine function approximated by the polynomials of its Taylor series.

References

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3 F. Diener, M. Diener (ed.), *Nonstandard Analysis in Practice*, Springer universitext, (1995), in particular Chapter 7: F. Koudjeti, I.P. van den Berg, *Neutrices, external numbers and external calculus*.

4 F. Koudjeti, *Elements of external calculus, with an application to mathematical finance*, Thesis, Groningen, Labyrint Publications, Capelle a/d IJssel, (1995).