Problemen

| Problem Section

Problem 21 (G.A. Kootstra)
Suppose that f, g are quadratic forms on $\mathbf{R}^n$, $n \geq 2$, such that $f(\underline{x}) = g(\underline{x}) = 0$ if and only if $\underline{x} = \underline{0}$. Show that there are real numbers a, b such that $af + bg$ is positive definite.

Problems 22 and 23 are due to Aad Thoen. In these problems an n-binary number stands for a number that, written in base n, consists of digits 0 and 1 only, ending with a 1. For example, 111001100000110101 is 5-binary number. It is also a square.

Problem 22
Prove that for any natural number N there are infinitely many 4-binary squares and 3-binary cubes with more than N digits equal to 1.

Problem 23 (Open problem)
Are there, for any natural number N, 3-binary squares with more than N digits equal to 1?

Solutions to volume 2, number 1 (March 2001)

Problem 15
Compute the surface area of the figure $|xy(x+y)| \leq 1$ in the plane.

Solution by Frits Beukers. The area is equal to the integral $A = \int dxdy$ taken over the set $|xy(x+y)| \leq 1$. Introduce the new variables ξ, η by $x = 1/\xi$ and $y = \eta/\xi$, to obtain

$$A = \int \frac{d\xi d\eta}{\xi^3},$$

where the integration is taken over the set $|\eta(\eta+1)| \leq |\xi|^3$. First integrate over ξ from $|\eta(\eta+1)|^{1/3}$ to ∞ and from $-|\eta(\eta+1)|^{1/3}$ to $-\infty$. After these integrations we are left with

$$A = \int_{-\infty}^{\infty} \frac{d\eta}{|\eta(\eta+1)|^{2/3}}.$$

This integral can be split into three parts, from $-\infty$ to -1, from -1 to 0 and from 0 to ∞. In the first part, substitute $\eta = -1/t$, in the second $\eta = -t$ and in the third $\eta = t/(1-t)$. In each case the integral becomes

$$\int_0^1 \frac{dt}{t^{2/3}(1-t)^{2/3}}.$$

This is an Euler beta-integral, and its value equals $\Gamma(1/3)^2/\Gamma(2/3)$. All three integrals together yield $3\Gamma(1/3)^2/\Gamma(2/3)$.

Problem 16
Show that for any given sequence $x_1, ..., x_n \in [0,1]$ there exists a sequence $y_1, ..., y_n \in [-1,1]$ such that $|y_i| = x_i$ and such that for each $k \leq n$

$$\left| \sum_{i=1}^{k} y_i - \sum_{j=k+1}^{n} y_j \right| \leq 2.$$

This problem is taken from *The ring loading problem*, by Schrijver, Seymour and Winkler, SIAM J. Discr. Math. 11 (1998), no. 1, 1–14. A general version of the problem, in which one chooses $y_i = x_i$ or $y_i = -x'_i$ for non-negative numbers $x_i + x'_i \leq 2$, remains open.

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

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Solution Think of the y_i as jumps of length x_i, to the right if $y_i > 0$ and to the left if $y_i < 0$. Define $p_k = \sum_{i=1}^{k} y_i$ as the position after k-jumps. Try this algorithm: let the k-th jump be to the right if $p_{k-1} \leq 0$; otherwise jump to the left. Then $p_k \in (-1, 1]$ so

$$\left|\sum_{i=1}^{k} y_i - \sum_{j=k+1}^{n} y_j\right| = |2p_k - p_n| \leq 2 + |p_n|,$$

which is sufficient only if we land at 0. Adapt the algorithm. For $a \in [0,1]$ jump to the right if $p_{k-1} \leq a$; otherwise jump to the left. Then $p_k \in (a-1, a+1]$ so we are done if we can show that $p_n = 2a$ for some a.
We vary the parameter a and denote the end-point by $p_n(a)$. Note that there is one ambiguity in the algorithm, which occurs if $p_k = a$ for some k. In that case we might just as well jump to the left, ending up at $2a - p_n(a)$ instead of $p_n(a)$. Both points are equidistant to a, so $a \to |a - p_n(a)|$ is a continuous function, with values in $[0,1]$. Now $p_n(a)$ is piecewise constant, thus the graph of $a \to |a - p_n(a)|$ has slope ± 1. More specifically, the slope is $+1$ if $p_n(a) < a$ and -1 if $p_n(a) > a$ and at the ambiguous points, the sign of the slope changes. By the intermediate value theorem, either $p_n(0) = 0$ or there exists an $a_0 > 0$ with $|a_0| = |a_0 - p_n(a_0)|$ and slope -1. In both cases we are done.

Problem 17

Given a triangle ABC with sides of length a, b, c. Three squares U, V, W with sides of length x, y, z, respectively, are inscribed in the triangle. The square U has two vertices on BC, one on AB and one on AC. In the same way, V and W have two vertices on AC and two vertices on AB, respectively. Find the minimal value of $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$.

Solutions by Ruud Jeurissen (Nijmegen), Kees Jonkers (Alkmaar), Hans Linders (Eindhoven), A.J.Th. Maassen (Milsbeek), Minh Can (Houston, 2 solutions). The minimal value is $3 + 2\sqrt{3}$. Can and Jonkers give a solution that uses Weitzenbock's inequality: if a, b, c are the lengths of the sides of a triangle and S is its area, then $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$. Can, Jeurissen, Linders and Maassen give similar solutions, as follows. The two vertices of V on BC divide this side into three parts with lengths $x \cot\beta$, x, and $x \cot\gamma$. Therefore $a = x\cot\beta + x + x\cot\gamma$ and so

$$\frac{a}{x} = 1 + \cot\beta + \cot\gamma.$$

We have similar expressions for b/y and c/z, and by adding them we obtain

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3 + 2(\cot\alpha + \cot\beta + \cot\gamma)$$

to be minimized under the conditions $\alpha + \beta + \gamma = \pi$ and positive α, β, γ. By Lagrange multipliers one finds that $\alpha = \beta = \gamma = \frac{\pi}{3}$.