Problemen

| Problem Section

Problem 15 (Frits Beukers)
Compute the surface area of the figure $|xy(x+y)| \leq 1$.

Problem 16 (Lex Schrijver)
Show that any given sequence $x_1, ..., x_n \in [0,1]$ there exists a sequence $y_1, ..., y_n \in [-1,1]$ such that $|y_i| = x_i$ and such that for each $k \leq n$:

$$\left| \sum_{i=1}^{k} y_i - \sum_{j=k+1}^{n} y_j \right| \leq 2.$$

Problem 17 (Alex Heinis)
Given a triangle ABC with sides of length a, b, c. Three squares V, W, Z with sides of length x, y, z, respectively, are inscribed in the triangle. The square V has two vertices on BC, one on AB and one on AC. In the same way, W and U have two vertices on AB and two vertices on AC, respectively. Find the minimal value of $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$.

Solutions to volume 1, number 3 (September 2000)

Problem 7
For a natural number n not divisible by 3 show that the only integer solutions of

$$(X^2 - YZ)^n + (Y^2 - XZ)^n + (Z^2 - XY)^n = 1$$

are the trivial ones.

Solution by the contributor of the problem, H. van den Berg. We may assume that $n > 3$ as the cases $n = 1$ and $n = 2$ are straightforward. Consider the symmetric polynomial

$$P(X, Y, Z) = X^2 + Y^2 + Z^2 - XY - XZ - YZ.$$

If we can show that $P(X, Y, Z)$ divides the symmetric polynomial $Q(X, Y, Z) = (X^2 - YZ)^n + (Y^2 - XZ) + (Z^2 - XY)^n$ then under the given conditions $P(X, Y, Z) = 1$. This implies that $(X - Y)^2 + (X - Z)^2 + (Y - Z)^2 = 2$. One of the three squares is zero and the other two squares are equal to one. Without loss of generality we may assume that $Y = X$ and $Z = X + 1$. By substitution in the original equation we find that the only solutions are the trivial ones.
It remains to show that $P(X, Y, Z)$ divides $Q(X, Y, Z)$. Suppose that A is a cyclic permutation matrix of order n, so $A^n = I$, and suppose that a, b, c are integers. The determinant of the matrix $B = aI + bA + cA^{-1}$ is equal to

$$\det B = a^n + b^n + c^n + k_1 a^{n-2} bc + k_2 a^{n-4} b^2 c^2 + \ldots \quad (1)$$

for integers $k_1, k_2, \ldots$ Now verify that, since n is not divisible by 3, $\det(I + A + A^{-1}) = 3$. By subsituting $a = b = c = 1$ in equation (1) we see that $k_1 + k_2 + k_3 + \ldots = 0$. Use this to rewrite the determinant as

$$\det B = a^n + b^n + c^n + k_1 a^{n-2}(bc - a^2) + k_2 a^{n-4}(b^2c^2 - a^4) + \ldots$$

So we have that $\det B = a^n + b^n + c^n + N(a^2 - bc)$ for some integer N. Choose $a = X^2 - YZ$ and $b = Y^2 - XZ$ and $c = Z^2 - XY$ and observe that $a^2 - bc$ is divisible by $X^2 + Y^2 + Z^2 - XY - XZ - YZ$. Add the columns of B to find that $a + b + c$ is an eigenvalue. Since for our choice $P(X, Y, Z) = a + b + c$, we conclude that $P(X, Y, Z)$ divides $\det B$.

Sinds mei 2000 werkt Joost van Hamel als postdoc aan de faculteit wiskunde van de Universiteit van Sydney. Vielen hem in de vorige twee afleveringen van deze column vooral de verschillen met Nederland op, nu springen hem allerlei overeenkomsten in het oog.

Problem 8

Partition the set of numbers $\{1, 2, \ldots, 2n\}$ into n pairs and denote the set of pairs by A. Define a partial order on A by $\{a_i, b_i\} < \{a_j, b_j\}$ if and only if $a_j < a_i < b_i < b_j$. As usual, we call a well ordered subset of $(A, <)$ a chain and we say that two chains A_1, A_2 cover A if $A_1 \cup A_2 = A$. Count the number of partitions into pairs for which A can be covered by two chains.

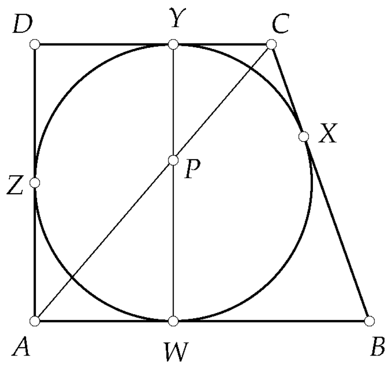
Solution This is a problem with a pitfall. It is tempting to count as follows. Suppose that A is a set of pairs covered by A_1 and A_2. Let $V_1 = \{v_1, v_2, \ldots, v_{2k}\}$ be the set of all elements of A_1. Then obviously the chain A_1 has to be $(v_1, v_{2k}) > (v_2, v_{2k-1}) > \ldots$. So to count all the partitions you just have to count all subsets of even order of $\{1, 2, \ldots, 2n\}$ and, by symmetry, divide by 2. This gives the answer 2^{2n-2}. There is a mistake in the argument, as a check of the case $n = 2$ gives 3 partitions into two pairs, all of which are covered by two chains. The right answer to the problem is $\binom{2n}{n}/2$. Ronald de Man has a short solution, but it is not elementary. A crisp counting argument would be appreciated.

Problem 9

Given a trapezoid with a right angle and an inscribed circle. Let L be the line through the intersection of the diagonals, parallel to the base of the trapezoid. Show that the length of the segment on L bounded by two sides is equal to the height of the trapezoid.

Solutions by L. Bleijenga (Den Haag), Minh Can (Washington DC), Kees Jonkers (Alkmaar), Aad Goddijn (Utrecht), W. Kleijne (Heerenveen), Floor van Lamoen (Goes), A.J. Tiggelaar (Leeuwarden).

Nearly all contributors supply a very similar solution as given below. Some contributors give more than one solution. Goddijn and Jonkers give a solution by Brianchon's theorem. Aad Goddijn and Floor van Lamoen remark that the problem can be generalized to quadrilaterals.



Let $ABCD$ be the given trapezoid, A and D being right angles, so that AD is the altitude, say of length h. The radius of the inscribed circle is thus $h/2$. Let $WXYZ$ be the points of contact of the inscribed circle with the trapezoid. Let P be the point of intersection of AC and WY. We see that P divides AC such that $AP : PC = AW : YC$. Let P' be the point of intersection of AC and ZX. We will show that $P' = P$.

Let Δ denote area. First note that $\Delta AP'Z : \Delta CP'X = P'A \cdot P'Z : P'C \cdot P'X$. Then, because angles PZA and PXC are supplementary (AD and BC being tangents to a circle at the ends of chord ZX), we also see that $\Delta AP'Z : \Delta CP'X = ZA \cdot ZP' : XP' \cdot XC$. We conclude $AP' : P'C = ZA : XC = AW : YC$ and we see that YW and ZX intersect in a point $P = P'$ on AC.

With the same type of reasoning we find that YW and ZX intersect on BD as well. So we have that YW, XZ, AC and BD are concurrent in a point Q. Now let the parallel line to AB through Q meet AD in R and BC in S. We have seen that $QR = h/2$. Since C and D are both at distance h from AB, we see that triangles ABD and ABC must intercept congruent segments of RS and we are done.

Solutions to some problems of yore

Below are the solutions to the problems 961–978, which belong to the Problem Section of the fourth series of Nieuw Archief.

Problem 961 (S. András and A. Bege)
Let x_1, x_2, x_3, x_4 be real numbers satisfying $0 < x_1 \leq x_2 \leq x_3 \leq x_4$ and let $s = x_1 + x_2 + x_3 + x_4$. Prove the following equality

$$\frac{x_1x_4}{s-x_3} + \frac{x_2x_1}{s-x_4} + \frac{x_3x_2}{s-x_1} + \frac{x_4x_3}{s-x_2} \leq \frac{x_1x_2}{s-x_3} + \frac{x_2x_3}{s-x_4} + \frac{x_3x_4}{s-x_1} + \frac{x_4x_1}{s-x_2}.$$

Solutions by R.A. Kortram, A.A. Jagers, H.J. Seiffert, A.J.Th. Maassen, F.J.H. Barning, J.H. van Geldrop, C. Jonkers. Most solutions note a small error in the original problem and are very similar. A.A. Jagers demonstrates that Maple can do all the work. Remove the denominators by multiplication, after which many terms cancel and by persistent computation we get the inequality $x_1x_2(x_1 + x_3) \leq x_2x_4(x_2 + x_4)$, which is an immediate consequence of $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$.

Problem 962
R.A. Kortram, A.A. Jagers, H.J. Seiffert, A.J.Th. Maassen and J.H. van Geldrop all observe that this problem is incorrect, although Seiffert manages to salvage the problem under an extra assumption.

Problem 963 (M.L.J. Hautus)
Let R denote a (not necessarily commutative) ring and let $a, b, c \in R$.
- Assume that $a^n + b^n = c^n$ for $n = 1, 2, 3$. Prove that this equation holds for all natural numbers n.
- Give an example of a ring for which the equation holds for $n = 1, 2$ but not for $n = 3$.

Solutions by R.A. Kortram, H.A. and R.W. van der Waall, A.A. Jagers, C. Praagman, A.J.Th. Maassen, J.H. Nieto, J.H. van Geldrop, R.H. Jeurissen. Below is the very efficient solution by R.H. Jeurissen. From $(a+b)^2 = a^2 + b^2$ we find $ab = -ba$ so $a^2b = ba^2$. From $(a^2 + b^2)(a + b) = (a + b)^3 = a^3 + b^3$ we find that $a^2b = -b^2a$. By induction we prove that $ab^n = -ba^n$ for all n:

$$ab^n = (ab)b^{n-1} = -bab^{n-1} = bba^{n-1} = b^a a^{n-2} = -a^2ba^{n-2} = ba^n.$$

Again by induction we now get the desidered equality $(a + b)^n = (a + b)(a + b)^{n-1} = (a+b)(a^{n-1} + b^{n-1}) = a^n + b^n + ab^{n-1} + ba^{n-1} = a^n + b^n$. As an example for the second question consider the quaternions (over any field). Take $a = i$ and $b = j$.

Problem 964 (J. van de Lune)
Let, as usual, $\pi(x)$ denote the numbers of primes not exceeding $x \in \mathbf{R}$ and let $\lfloor . \rfloor$ be the integer part. Prove that there exists a unique sequence $a(i)$ in $\mathbf{Z}$ such that $\pi(x) = \sum_{n \leq x} a(n) \lfloor \frac{x}{n} \rfloor$, $x > 0$. Determine this series and rewrite $\sum_{n=1}^{\infty} a_n / n^s$ in terms of 'well-known' Dirchlet series.

Solutions by K.W. Lau, M.T. Kosters, H.J. Seiffert. The solutions are the same. Let f be the characteristic function of the primes. Since

$$\pi(x) = \sum_{n \leq x} a(n) \lfloor \frac{x}{n} \rfloor = \sum_{n \leq x} a(n) \sum_{m \leq x/n} 1 = \sum_{m \leq x} \sum_{n|m} a(n),$$

it follows that $f(m) = \sum_{n|m} a(n)$. Apply Möbius inversion to find $a(n) = \sum_{p|n} \mu(\frac{n}{p})$, where μ is the Möbius function. Some calculation shows that for Re $s > 1$ we have that

$$\sum_{n=1}^{\infty} a(n)/n^s = \frac{1}{\zeta(s)} \sum_{p \text{ prime}} 1/p^s.$$

Problem 965 (J. van de Lune)
For $k \in \mathbf{N}$ let $M(k)$ denote the least common multiple of the integers $1, 2, \ldots, k$. Find the first $2k+1$ indices a_i in the continued fraction expansion

$$M(k)\log(1+1/M(k)) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Solution by F.J.M. Barning, who shows that $a_{2i+1} = 2i+1$ for $i = 0, \ldots, k$ and $a_{2i} = 2M(k)/i$ for $i = 1, \ldots, k$, for which he gives a self-contained proof. Van de Lune's original solution contains a short cut using Lambert's general continued fraction expansion of $\log(1+x)$ (Perron, Die Lehre von den Kettenbrüchen, (1929), p. 349).

Problem 966 (J.B. Melissen)
Show that for $0 < x < \pi$ we have $(\cos \frac{x}{2})^4 < (\frac{\sin x}{x})^3 < (\cos \frac{x}{\sqrt{5}})^5$.

Solutions by R.A. Kortram, A.A. Jagers, K.W. Lau, H.J. Seiffert, F.J.M. Barning, P. McCartney. Below is the solution by A.A. Jagers. The Bernoulli numbers B_n are defined by $\sum B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}$ for $|z| < 2\pi$. Then $z \cot z = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} z^{2n}$ for $|z| < \pi$ and hence by integration of $\cot z - z^{-1}$, recalling that $(-1)^n B_{2n} < 0$ for all $n > 0$

$$\log \frac{\sin z}{z} = -\sum_{n=1}^{\infty} \frac{4^n |B_{2n}|}{2n(2n)!} z^{2n} \qquad (|z| < \pi).$$

By using the duplication formula $\sin 2z = \sin z \cos z$ it follows that

$$\log \cos z = -\sum_{n=1}^{\infty} \frac{4^n(4^n - 1)|B_{2n}|}{2n(2n)!} z^{2n} \qquad (|z| < \pi/2).$$

Using these two equations it is easily verified that the logarithmic version of the inequality holds even term-wise

$$4 \log \cos \frac{x}{2} < 3 \log \frac{\sin x}{x} < 5 \log \cos \frac{x}{\sqrt{5}} \qquad (0 < x < \pi).$$

Problem 967 (F. Rothe)
Let$P_{2n+1} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ be the $2n+1$-th order MacLaurin polynomial of the function $\sin x$. Let c_{2n+1} be the number of real zeroes of P_{2n+1} counting multiplicities. Determine the limit

$$\lim_{n \to \infty} \frac{c_{4n+1}}{4n+1}.$$

Solutions by R.A. Kortram and A.A. Jagers. This was certainly one of the harder problems, requiring several steps to get to the final solution. F. Rothe needs twelve lemmas to get to the result. The solutions of R.A. Kortram and A.A. Jagers are somewhat shorter. Below is an outline of Kortram's solution, which consists of three parts.
STEP 1. If $0 < \alpha < 1$, then P_{8n+3} and $\sin x$ have the same number of zeroes on the disc $D(0, \frac{8n}{e}(1-\alpha))$ for sufficiently large n. For $|z| = \frac{8n}{e}(1-\alpha)$ we have that

$$|\sin z - P_{8n+3}(z)| = \Big|\sum_{k=4n+2}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}\Big| \le \sum_{k=8n+5}^{\infty}\frac{|z|^k}{k!}$$
$$\le \sum_{k=8n+5}^{\infty}\Big(\frac{8n(1-\alpha)}{k}\Big)^k < \sum_{k=8n}^{\infty}(1-\alpha)^k = \frac{1}{\alpha}(1-\alpha)^{8n}.$$

For n sufficiently large, we have that $|\sin z - P_{8n+3}(z)| < |\sin z|$ on the circle with radius $\frac{8n}{e}(1-\alpha)$, so the assertion follows from Rouché's theorem.

STEP 2. All zeroes of P_{8n+3} are real. We consider only positive real numbers x. If $x^2 < (8n+6)(8n+7)$ then the sequence $\frac{x^{8n+3+2k}}{(8n+3+2k)!}$ is decreasing for $k = 1, 2, 3, \ldots$. Thus we have that
$$\frac{x^{8n+5}}{(8n+5)!} - \frac{x^{8n+7}}{(8n+7)!} < \sin x - P_{8n+3} < \frac{x^{8n+5}}{(8n+5)!}.$$

In particular if $x < \frac{8n}{5}$, then $\sin x - P_{8n+3} < (\frac{8n}{8n+5})^{8n+5} < \frac{1}{e^5}$. So it appears that the graphs of sin and P_{8n+3} differ less than 1 and sin is always bigger. Every interval $(2k\pi, (2k+1)\pi)$ contains at least two zeroes for P_{8n+3} and therefore the number of real zeroes is at least $\frac{8n}{\pi e}(1-\alpha)$ in the disc. Step 1 gives that, asymptotically, this is the total number of zeroes of P_{8n+3} in the disc.

STEP 3. P_{8n+3} has no zeroes outside the disc $D(0, \frac{8n}{e}(1+\alpha))$.
Subsitute $x_0 = \frac{8n}{e}(1+\alpha)$ in inequality (1) to get that
$$\sin x_0 - P_{8n+3} \ge c\frac{(1+\alpha)^{8n+5}}{8n+5} > 2$$
for sufficiently large n. Since $\sin x - P_{8n+3}$ is increasing this implies that $P_{8n+3} < 0$ for $x > x_0$. Hence there are no real zeroes for $|x| \ge \frac{8n}{e}(1+\alpha)$. The number of zeroes is at most $\frac{8n}{\pi e}(1+\alpha)$. Combining this with our previous estimate we conclude that $\lim_{n\to\infty}\frac{c_{4n+1}}{4n+1} = \frac{2}{\pi e}$.

Problem 968 (H.J. Seiffert)
Let $f\colon [0,1] \to (0,\infty)$ be a C^1 function such that $\log f$ is convex. For real $p \neq 0$ define $S_p = \frac{1}{2}(f(0)^p + f(1)^p)$ and $I_p = \int_0^1 f(x)^p dx$. Show that $\frac{p+1}{p}I_{p+1} \le \frac{1}{p}S_pI_1 + S_1I_p$.

Solutions by R.A. Kortram and J.H. van Geldrop. Below is the solution of R.A. Kortram. We may assume that $f(0) = 1$. Define for $t \in [0,1]$
$$g(t) \;= \frac{(f^p(t)+1)}{p}\int_0^t f(x)dx + (f(t)+1)\int_0^t f^p(x)dx - \frac{2(p+1)}{p}\int_0^t f^{p+1}(x)dx.$$

It is immediate that $g(0) = 0$. Observe that the problem is to show that $g(1) \ge 0$ and we will do this by showing that g has a non-negative derivative.
$$g'(t) \;= \Big(\frac{f'}{f} - \frac{f-1}{\int_0^t f}\Big)f^p\int_0^t f + \frac{1}{p}\Big(\frac{(f^p)'}{f^p} - \frac{f^p-1}{\int_0^t f^p}\Big)f\int_0^t f^p.$$

Since $\log f$ is convex, so is $\log f^p$ and we have that both $\frac{f'}{f}$ and $\frac{(f^p)'}{f^p}$ are non-decreasing. Now the required follows from the observation that $\frac{g(t)-1}{\int_0^t g} = \frac{g'(\theta)}{g(\theta)}$ for some $\theta < t$ by the second mean value theorem.

Problem 969 (H.J. Seiffert)
Let a and b be positive and unequal real numbers. Prove that $\big(\frac{a^a}{b^b}\big)^{1/(a-b)} < (\sqrt{a}+\sqrt{b})^2$.

Solutions by F.J.M Barning, R.A. Kortram, A.A. Jagers, M.T. McGregor, K.W. Lau, C. van Berkel, A.J.Th. Maassen, J.H. Nieto, J.H. van Geldrop, R.H. Jeurissen, P. McCartney. Below is the solution by Nieto. By symmetry we may assume that $a > b > 0$. For $t > 1$ and $x > 0$ the inequality $(1+x)^t > 1+tx$ holds. Subsitute $t = a/b$ and $x = \sqrt{b/a}$ to obtain that $(1+\sqrt{b/a})^{a/b} > 1+\sqrt{a/b}$. Rewrite this to $(\sqrt{a}+\sqrt{b})^{2(a-b)} > a^a/b^b$ which gives the desired result after raising to the power $1/(a-b)$.

Oplossingen | Solutions

Problem 970 (H. Alzer)
Let a_i and p_i be positive numbers for $i = 1, \dots, n$ and $n \geq 2$, satisfying either $p_i \leq 1$ for all i, or $p_i \geq 1$ for all i. Prove that

$$\sum_{k=1}^{n} p_k a_k \leq \max_{1 \leq k \leq n} \Big(a_k \prod_{i=1}^{n} p_i + \sum_{i=1, i \neq k}^{n} a_i \Big).$$

When does equality hold?

Solutions by J.H. van Geldrop, G.W. Veltkamp, A.A. Jagers, H.J. Seiffert. Below is the efficient solution by Seiffert. Under the given conditions we have that

$$p_k - 1 \leq \Big(\prod_{i=1}^{k-1} p_i \Big)(p_k - 1)$$

where the empty product is understood to be 1. Summing over all $k = 1, \dots, n$ gives

$$\sum_{k=1}^{n} (p_k - 1) \;\leq \prod_{i=1}^{n} p_i - 1,$$

with equality only if all p_i are equal to 1. Take m to be the minimum of all a_i in case all $p_i \leq 1$ and the maximum of all a_i in case all $p_i > 1$. We obtain from the previous inequality that

$$\sum_{k=1}^{n} (p_k - 1) a_k \;\leq m \sum_{k=1}^{n} (p_k - 1) \leq m \Big(\prod_{i=1}^{n} p_i - 1 \Big),$$

which is equivalent to the desired inequality. There is equality if and only if for all but one index we have $p_i = 1$ and for the remaining index $a_k = m$.

Problem 971 (H. Alzer)
Let x and y be distinct positive real numbers amd r a nonnegative real number. Define

$$L_r(x,y) = \left[\frac{x^r - y^r}{r(x-y)} \right]^{1/(r-1)}, \qquad (r \notin \{0,1\})$$
$$L_0(x,y) = \frac{x-y}{\log(x/y)},$$
$$L_1(x,y) = \frac{1}{e} \left(\frac{x^x}{y^y} \right)^{1/(x-y)}.$$

Prove that for arbitrary nonnegative numbers r, s, t satisfying $r < s < t$ we have that

$$L_r(x,y)^{t-s} L_t(x,y)^{s-r} \; < L_s(x,y)^{t-r}.$$

Solutions by G.W. Veltkamp and A.A. Jagers. Below is the efficient solution by Jagers. We have to show that $L_r(x,y)$ is strictly concave in r. Let $x/y = e^{2u}$ and define $f(0) = 0$ and $f(r) = \log \frac{\sinh ru}{ru}$ for $r \neq 0$. Then

$$\log L_r(e^u, e^{-u}) \;= \frac{f(r) - f(1)}{r-1} = \frac{1}{r-1} \int_1^r f'(t) dt$$

and $L_r(x,y) = \sqrt{xy} L_r(e^u, e^{-u}$. Hence it suffices to show that f' is strictly concave on $[0, \infty)$. Consider the function $g(s) = \cosh(s) - \left(\frac{\sinh s}{s}\right)^3$, for which it is straightforward to verify that $g(s) \leq 0$. Then f satisfies the differential equation $u^{-3} \sinh^3(ut) f'''(t) = 2g(ut)$ which completes the proof.