Jan Aarts
Technische Universiteit Delft
Faculteit Informatie Technologie en Systemen
Afdeling Mediamatica
Postbus 5031, 2600 GA Delft
j.m.aarts@its.tudelft.nl

Robbert Fokkink
Technische Universiteit Delft
Faculteit Informatie Technologie en Systemen
Afdeling Mediamatica
Postbus 5031, 2600 GA Delft
r.j.fokkink@its.tudelft.nl

Godfried Kruijtzer
Vlietwyck 24, 2271 EV Voorburg
Technische Universiteit Delft
Faculteit van Civiele Techniek en Geowetenschappen

Morphic numbers

Benediktijner monnik en architekt Dom Van der Laan heeft een monnikenleven lang besteed aan het uitwerken van een eigen maatsysteem dat kenmerkend is voor de stijl van wat de 'Bossche School' is gaan heten. Basis van dit maatsysteem is het Plastische Getal, door Van der Laan uitgevonden omdat hij de gulden snede te beperkt vond. Het plastisch getal en de gulden snede zijn beide zogenaamde 'morphic numbers'. Of er nog meer van dergelijke getallen bestaan bleek niet bekend te zijn.

We define morphic numbers and show that there are only two such numbers, namely the divine proportion and the plastic number of Dom Van der Laan.

Introduction

A natural scale for the comparison of the magnitude of numbers is an arithmetic sequence of which the set of integers with the natural order is the prototype. For comparing geometric measures or the extent of objects in three space however, a geometric sequence

$$\ldots, p^{-2}, p^{-1}, 1, p, p^2, \ldots \quad (p > 1)$$

is a more natural scale. When this scale is used the relative difference of two consecutive numbers on the scale, which is defined as the quotient $(p^{k+1} - p^k)/(p^{k+1} + p^k)$, does not depend on the value of k. If p is the divine proportion, [1], the geometric sequence has two additional interesting properties. First, the sum of any two consecutive elements of the sequence is equal to the next element of the sequence or, equivalently, $1 + p = p^2$. When this scale is used for marking the length of objects the property entails that the juxtaposition of objects, one object next to the other, of length p^k and p^{k+1} respectively produces an object of length p^{k+2}. Secondly, the difference of any two consecutive numbers in the sequence is the previous number in the sequence or, equivalently, $p - 1 = p^{-1}$. Note that if intervals with a common endpoint of length 1 and p are placed alongside each other (in superposition) then the excess of the longer interval over the shorter is $p - 1$, see figure 1.

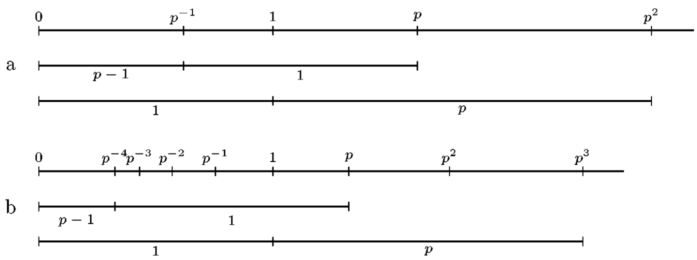


Figure 1. A system of measures is a geometric sequence with ratio p. There are two extra conditions, namely, $p+1$ (superposition of two consecutive measures) and $p-1$ (juxtaposition of two consecutive measures) are measures of the system. The only possible measure systems are based on the golden number (a) and the plastic number (b).

These properties of the divine proportion suggest the following definition.

Definition. *A real number $p > 1$ is called a* morphic number *if there exist natural numbers k and l such that*

$$p + 1 = p^k \quad \textit{and} \quad p - 1 = p^{-l}.$$

One might wonder whether there exist morphic numbers apart from the divine proportion. In [3] Dom Hans van der Laan, architect and member of the Benedictine order, introduced the *plastic number* as the ideal ratio of the geometric scale for spatial objects. The plastic number is the real solution of the equation $x^3 - x - 1 = 0$, see [2]. As

$$x^5 - x^4 - 1 = (x^3 - x - 1)(x^2 - x + 1),$$

it follows that the plastic number also satisfies $x - 1 = x^{-4}$ and thus is a morphic number (with $k = 3$ and $l = 4$). According to Padovan [4, p. 96], at about the same time the same discovery was made by Gérard Cordonnier, a French architecture student. It was conjectured in [2] that the divine proportion and the plastic number of Dom Van der Laan are the only morphic numbers. Theorem 1 states that the conjecture holds true.

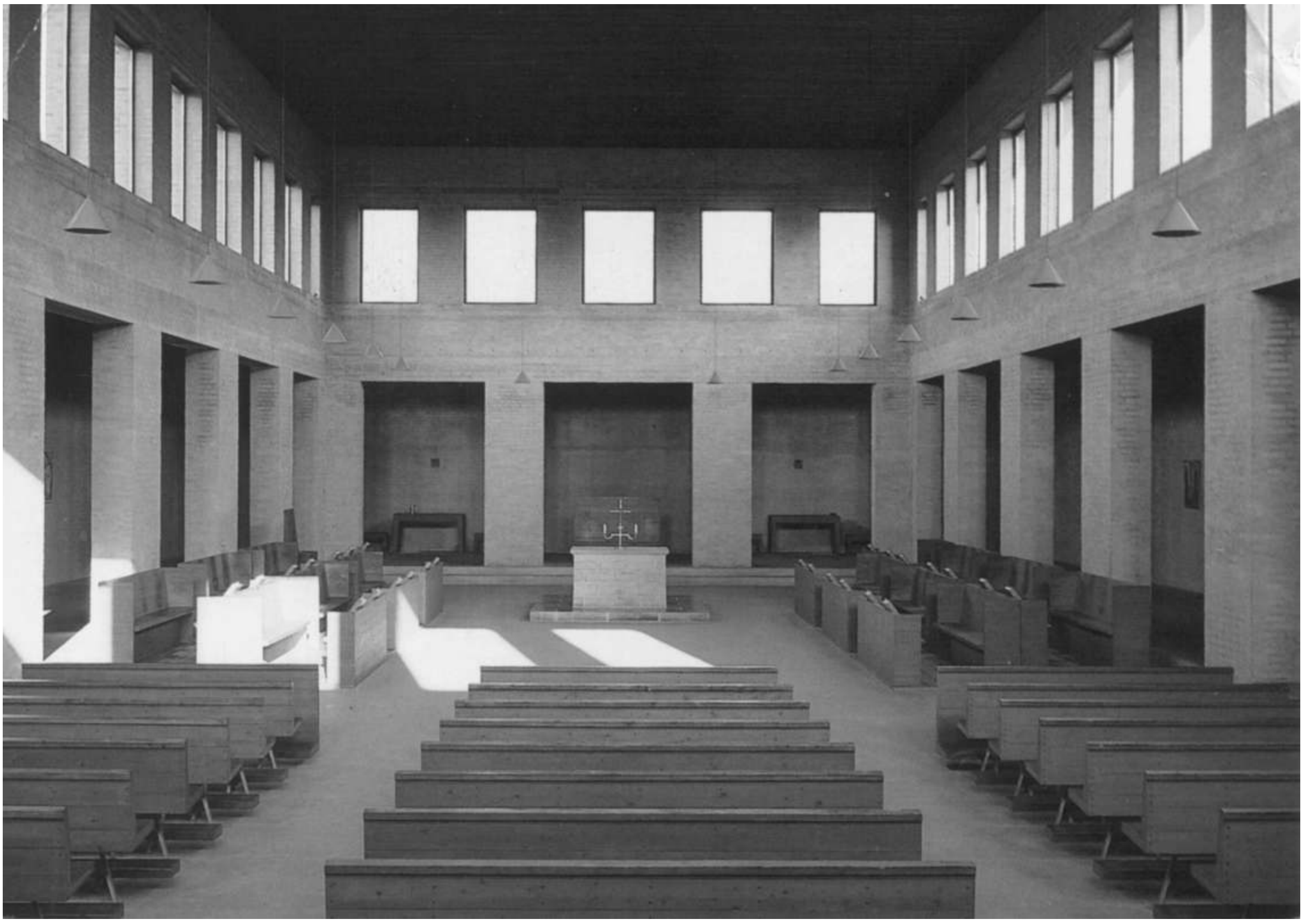
Trinomials

A morphic number satisfies two equations which can be written in the following form

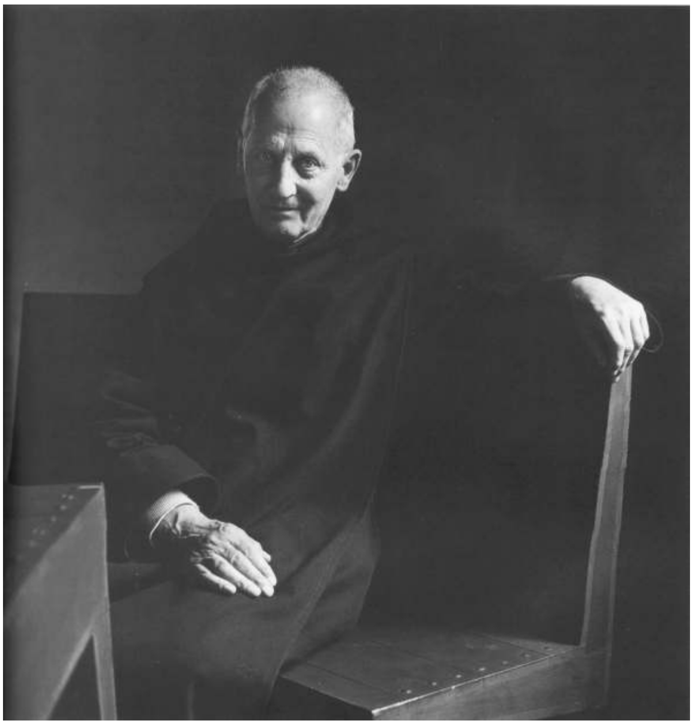
$$x^n - x - 1 = 0 \quad \text{and} \quad x^m - x^{m-1} - 1 = 0, \quad n, m \geq 2.$$

The left-hand sides of these equations are polynomials. In this particular case of polynomials consisting of three terms only the polynomials are called *trinomials*. Two mathematicians from the University of Bergen (Norway), Ernst Selmer and Helge Tverberg, studied trinomials some forty years ago [5, 6]. As we shall demonstrate below, their results imply that there exists no morphic number apart from the two that we know already, thus confirming the intuitive argument of [2].

The trinomials $X^n - X - 1$, $n \geq 3$, were studied by Selmer [5]. He showed that these trinomials are irreducible, i.e., no such trinomial has a polynomial factor (with integer coefficients) apart from 1 and itself. Tverberg extended Selmer's result for trinomials of the form $X^m \pm X^k \pm 1$, $m \geq 3$ and $1 < k < m$, in [6]. Tverberg showed that such a trinomial is either irreducible or a product of two polynomials; in the latter case the factorization can be arranged in such a way that the first polynomial is irreducible (or possibly a constant) and the second polynomial



The interior of the church of the abby 'Sint Benedictusberg' at Mamelis (Vaals, The Netherlands). The church is build according to the architectural principles of Dom Van der Laan.



The Benedict monk, priest and architect Dom Hans van der Laan (1904-1991) designed the so-called architectonic space. His basic building block of the environment is the plastic number.

has only zeros of modulus 1. The above results concerning the factors of trinomials apply directly to the morphic numbers. A morphic number is a common root of two polynomials and its existence entails a common polynomial factor (with integer coefficients).

Lemma. *Each trinomial $X^m - X^{m-1} - 1$, $m \geq 3$, is either irreducible or the product of an irreducible polynomial and the trinomial $X^2 - X + 1$.*

Proof. If the trinomial is reducible then by Tverberg's result it is the product of an irreducible polynomial and a polynomial all zeros of which have modulus 1. Note that the zeros of the trinomial satisfy the equation

$$|x|^{m-1} = \frac{1}{|x-1|}.$$

Thus, if a zero ξ has modulus 1, then ξ is a complex number whose distances both to the origin and to the (real) number 1 are equal to 1. It follows that $\xi = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$. If these zeros occur, then the trinomial is divisible by $X^2 - X + 1$. □

Now we can prove the main result of this note.

Theorem. *There exist only two morphic numbers, namely the divine proportion and the plastic number of Dom Van der Laan.*

Proof. A morphic number is a common zero of the trinomials $X^n - X - 1$ and $X^m - X^{m-1} - 1$, where $n, m \geq 2$. For $n = m = 2$ the trinomials are the same and the zeros are the divine proportion and minus its inverse. So we may assume that $n, m \geq 3$. In order that a common zero exists the trinomials must have a common factor. As the first trinomial is irreducible, this can only occur if the first trinomial divides the second. In view of the lemma, a necessary condition for the existence of a common zero is that there is the factorization

$$(X^n - X - 1)(X^2 - X + 1) = X^m - X^{m-1} - 1.$$

By direct computation one may verify that $n = 3$ and $m = 5$. And in that case the solution of the equations is the plastic number of Dom Van der Laan. □

Acknowledgements
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