Lutz Mattner
Department of Statistics
University of Leeds
Leeds, LS2 9JT, England
mattner@amsta.leeds.ac.uk

Complex differentiation under the integral

We present a theorem and corresponding counterexamples on the classical question of differentiability of integrals depending on a complex parameter. The results improve on the ones usually given in textbooks.

The following theorem on complex differentiation under the integral might be the most convenient of its kind, fits well in a course on real and complex analysis, and appears to be little known.

Theorem. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, let $G \subset \mathbf{C}$ be open, and let $f : G \times \Omega \to \mathbf{C}$ be a function subject to the following assumptions:*

A_1: *$f(z, \cdot)$ is $\mathcal{A}$-measurable for every $z \in G$,*
A_2: *$f(\cdot, \omega)$ is holomorphic for every $\omega \in \Omega$,*
A_3: *$\int |f(\cdot, \omega)| \, d\mu(\omega)$ is locally bounded, that is,*

$$z_0 \in G \implies \exists\, \delta > 0 : \quad \sup_{z \in G, |z - z_0| \leq \delta} \int |f(z, \omega)| \, d\mu(\omega) < \infty. \quad (1)$$

Then $\int f(\cdot, \omega) \, d\mu(\omega)$ is holomorphic and may be differentiated under the integral. More precisely, we have for every $n \in \mathbf{N}_0$ the following conclusions:

C_1: *$\partial_z^n f$ is $\mathcal{B}(G) \otimes \mathcal{A}$-measurable and, for every $A \subset G$, $\sup_{z \in A} |\partial_z^n f(z, \cdot)|$ is $\mathcal{A}$-measurable,*

C_2: *we have the implication*

$$K \subset G \text{ compact} \implies \int \sup_{z \in K} |\partial_z^n f(z, \omega)| \, d\mu(\omega) < \infty, \quad (2)$$

C_3: *$\int f(\cdot, \omega) \, d\mu(\omega)$ is holomorphic in G with*

$$\partial_z^n \int f(z, \omega) \, d\mu(\omega) = \int \partial_z^n f(z, \omega) \, d\mu(\omega) \qquad (z \in G). \quad (3)$$

Notation. We write $\mathbf{N} := \{1, 2, \ldots\}$ and $\mathbf{N}_0 := \{0\} \cup \mathbf{N}$. If a set $\mathcal{X}$ is naturally equipped with a metric or topology, as in the case of $\mathcal{X} = \mathbf{C}$ or $\mathcal{X} = G$ in the theorem, then $\mathcal{B}(\mathcal{X})$ denotes the Borel σ-algebra on $\mathcal{X}$, that is, the σ-algebra generated by the open sets, and measurability refers to this σ-algebra. The product σ-algebra of two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Assumptions A_1 and A_2 are the obvious ones for (3) to make sense at all. But in place of A_3, the natural if somewhat optimistic assumption might appear to be

A_3^-: $\int |f(z, \omega)| \, d\mu(\omega) < \infty \quad (z \in G)$.

Counterexample 1, given below, shows that under assumptions A_1, A_2, and A_3^-, and even when $\Omega = \mathbf{N}$ and μ is counting measure, the function $F := \int f(\cdot, \omega) \, d\mu(\omega)$ can be discontinuous and hence nonholomorphic (take $a_0 = 1$). Alternatively, F can be holomorphic with $\int \partial_z f(\cdot, \omega) \, d\mu(\omega)$ well defined and discontinuous (take $a_0 = 0$ and $a_1 = 1$).

Counterexample 1. *Let $(a_n : n \in \mathbf{N}_0)$ be a sequence of complex numbers. Then there exists a sequence of polynomials, $(p_\nu : \nu \in \mathbf{N})$, such that*

$$\sum_{\nu \in \mathbf{N}} |p_\nu^{(n)}(z)| < \infty \qquad (n \in \mathbf{N}_0,\, z \in \mathbf{C}), \quad (4)$$

$$\sum_{\nu \in \mathbf{N}} p_\nu^{(n)}(z) = a_n \cdot 1_{\{0\}}(z) \qquad (n \in \mathbf{N}_0,\, z \in \mathbf{C}). \quad (5)$$

Here we have used the alternative notation $f^{(n)}$ for the nth derivative $\partial_z^n f$, and have written 1_A for the indicator (or characteristic function) of a set A.

What happens if we assume, in addition to A_1, A_2, and A_3^-, that $\int f(\cdot, \omega) \, d\mu(\omega)$ converges locally uniformly? For abstract $(\Omega, \mathcal{A}, \mu)$, it is not clear what this means, so let us think of the case of counting measure on $\mathbf{N}$. There the assumption means the locally uniform convergence of the sequence of the partial sums $\sum_{\nu=1}^N f(\cdot, \nu)$. This would imply locally uniform convergence of the derivatives $\sum_{\nu=1}^N \partial_z^n f(\cdot, \nu)$, so that an example as in (5), with $f(\cdot, \nu)$ in place of p_ν and with some $a_n \neq 0$, would be impossible. Nevertheless, conclusion C_3 may fail due to nonexistence of the right hand side in (3):

Counterexample 2. *There exists a sequence of polynomials, $(p_\nu : \nu \in \mathbf{N})$, such that*

$$\sum_{\nu = 1}^{\infty} p_\nu \qquad \text{converges locally uniformly in } \mathbf{C}, \quad (6)$$

$$\sum_{\nu \in \mathbf{N}} |p_\nu(z)| < \infty \qquad (z \in \mathbf{C}), \quad (7)$$

$$\sum_{\nu \in \mathbf{N}} |p'_\nu(0)| = \infty. \quad (8)$$

It is no accident that the series of the counterexamples behave badly only in a rather small subset of G, here consisting of just one point.

Side remark. *If, in the theorem, assumption* A_3 *is replaced by* A_3^-*, then the conclusion remains valid with some dense open subset* $G^* \subset G$ *in place of* G.

Since the conclusion of the side remark is too weak for most applications encountered in the daily life of an analyst, the counterexamples show the need for finding some convenient assumption stronger than A_3^-. For this problem, the best textbook treatments known to me are given by Dieudonné [6, § 13.8.6.1], Elstrodt [8, p. 147], and Königsberger [12, p. 283]. They have the conclusion C_3 under the assumptions A_1, A_2, and

$$A_3^+\colon\ z_0 \in G \implies \exists\, \delta > 0 : \int \sup_{z \in G, |z-z_0| \le \delta} |f(z,\omega)|\, d\mu(\omega) < \infty.$$

Obviously, assumption A_3^+ alone is more restrictive than assumption A_3. On the other hand, conclusion C_2 of the theorem shows that the two conditions are in fact equivalent in the presence of assumptions A_1 and A_2. Hence the point of the present theorem is not greater generality, but greater convenience in checking the assumptions. The following simple example illustrates this.

Example (holomorphic Fourier transforms). *Let* $g : \mathbf{R} \to \mathbf{C}$ *be locally integrable with respect to Lebesgue measure, let*

$$H := \left\{ z \in \mathbf{C} : \int |e^{izt} g(t)|\, dt < \infty \right\},$$
$$\widehat{g}(z) := \int e^{izt} g(t)\, dt \qquad (z \in H),$$

and assume that $G := \overset{\circ}{H}$*, the interior of* H*, is not empty. Then* $\widehat{g}$ *is holomorphic in* G *and may be differentiated under the integral.*

Proof. We apply the theorem to $\Omega := \mathbf{R}$ with Lebesgue measure, and $f(z,t) := g(t)e^{izt}$ for $z \in G$ and $t \in \mathbf{R}$. Assumptions A_1 and A_2 are obviously fulfilled. To check A_3, it suffices to observe that $\int |f(x+iy,t)|\, dt = \int e^{-yt} |g(t)|\, dt$ depends only on y and is a convex $[0,\infty]$-valued function of this real variable (in particular, G is a strip of the form $\mathbf{R} + iI$ with $I \subset \mathbf{R}$ an open interval), and hence surely is locally bounded on G as a function of the complex variable $z = x + iy$. □

Of course, one may also directly check assumption A_3^+ in this example, using

$$\sup_{z \in G, |z-z_0| \le \delta} |f(z,t)| \le \left(e^{-(y_0+\delta)t} + e^{-(y_0-\delta)t} \right) \cdot |g(t)|.$$

The following result shows that it is still possible to weaken assumption A_3 a bit without invalidating the theorem. Again, this does not achieve greater generality, but may afford greater convenience in checking the assumptions. In this case, however, I am not aware of any natural application.

Addendum to the theorem. *The theorem remains valid if assumption* A_3 *is replaced by*

A_3^0*:* $\int |f(\cdot,\omega)|\, d\mu(\omega)$ *is locally integrable with respect to Lebesgue measure on* G*, that is,*

$$z_0 \in G \implies \exists\, \delta > 0 \text{ such that:} \tag{9}$$
$$\iint_{z = x+iy \in G,\, |z-z_0| \le \delta} \int |f(z,\omega)|\, d\mu(\omega)\, dx dy < \infty.$$

Literature

Let us first consider the very special case in which μ is counting measure on $\Omega = \mathbf{N}$. Then the assumption A_1 and the conclusion C_1 of the theorem are trivial, and the implication 'A_2, $A_3 \implies C_2$ with $n = 0$' is implicitly given by Remmert in [17, § 8.4.4], but not in [18], crediting Martin Reinders. (The proof of conclusion C_2 in the proof of the theorem below is analogous to the argument of Reinders.) Of course, the conclusions 'C_2 with n arbitrary' and C_3 then follow by standard theorems on series of holomorphic functions [17 or 18, § 8.4.2].

For more general measure spaces, Światkowski [22] comes close to formulating and proving the theorem of the present paper. (Without Burckel [2], I would probably not have found this reference.) His Lemma on page 63 has for Ω the unit interval with Lebesgue measure, but no special property of this measure space except σ-finiteness is used. His claim amounts to: 'A_1, A_2, $A_3 \implies C_3$'. He neither proves nor explicitly assumes the joint measurability of f (contained in our conclusion C_1), but uses it implicitly in his proof via Fubini, which hence might be regarded as incomplete. (Concerning the necessity of the product measurability assumption in Fubini's theorem, one may consult [15].)

The same implication 'A_1, A_2, $A_3 \implies C_3$' is stated and proved by Everitt, Hayman & Nasri-Roudsari [10], apparently independently of [22]. They assume μ to be σ-finite and prove the needed product measurability before applying Fubini's theorem. The theorem of the present paper improves on this by adding conclusion C_2 and omitting the assumption of σ-finiteness.

Counterexamples proving 'A_1, A_2, $A_3^- \not\Rightarrow \int f(\cdot,\omega)\, d\mu(\omega)$ holomorphic' were supplied by Światkowski [22] and Hayman [11]. The latter example is studied in greater detail and generality by Everitt & Hayman [9] and by Chen & Hayman [4]. The counterexamples given in the present paper are designed to show that something can go wrong even under natural additional conditions. The construction of Counterexample 1 refines one presented by Remmert [19 or 20, § 12.3.1]. The construction of Counterexample 2 slightly modifies an unpublished one of Saeki, see Burckel [3], which is mentioned without details in [17, § 3.3.2], but not in [18]. In Saeki's example, one has polynomials p_ν satisfying (6), (7), and $\sup_{|z| \le \epsilon} \sum_{\nu \in \mathbf{N}} |p_\nu(z)| = \infty$ for every $\epsilon > 0$. In Counterexample 2, the third of these properties is replaced by (8), which, in view of the theorem, is more restrictive.

A theorem analogous to our side remark is also proved by Światkowski [22, p. 64]. This kind of result and its method of proof goes back to Osgood [16], see [2, p. 227].

Proofs

The following known lemma is used below to prove the measurability statements of the theorem. Versions of it have been discussed in dozens of journal articles, starting with Lebesgue [13, seventh page]. Many later references are given by Averna [1]. Nevertheless, most accounts of measure and integration theory ignore the lemma, or present versions insufficiently general. Hence we include a proof here.

Lemma. *Let* $(\mathcal{X}, \mathcal{A})$ *be measurable space, let* $(\mathcal{Y}, \tau)$ *be a topological space with countable basis, and let* $(\mathcal{Z}, d)$ *be a metric space. Let further* $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ *be a function such that* $f(\cdot, y)$ *is measurable for every* $y \in \mathcal{Y}$ *and* $f(x, \cdot)$ *is continuous for every* $x \in \mathcal{X}$*. Then* f *is* $\mathcal{A} \otimes \mathcal{B}(\mathcal{Y})$*-measurable.*

Proof. Let $(U_n : n \in \mathbf{N})$ be an enumeration of a basis of τ. For each $n \in \mathbf{N}$, let $\{E_{n,1}, \dots, E_{n,k(n)}\}$ denote the partition of $\mathcal{Y}$ generated by $\{U_1, \dots, U_n\}$. Choose $y_{n,i} \in E_{n,i}$. By the measurability of the $E_{n,i}$ and the measurability of the $f(\cdot, y)$, the function $f_n : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, defined by

$$f_n(x,y) := f(x, y_{n,i}) \qquad (y \in E_{n,i}),$$

is measurable. Further, using the continuity of each $f(x,\cdot)$, it is easily seen that $\lim_{n\to\infty} f_n = f$ pointwise. (In detail: Let $(x,y) \in \mathcal{X} \times \mathcal{Y}$ and let V be a neighbourhood of $f(x,y)$. Then there is a neighbourhood U of y with $f(x,U) \subset V$. We have $y \in U_{n_0} \subset U$ for some $n_0 \in \mathbf{N}$. Hence, for every $n \geq n_0$, there is an $i(n) \in \{1, \dots, k(n)\}$ with $y \in E_{n,i(n)} \subset U$, and hence $f_n(x,y) = f(x, y_{n,i(n)}) \in f(x,U) \subset V$.) It follows that f is measurable. (For this last step, see [7, Theorem 4.2.2]. In the special case $\mathcal{Z} = \mathbf{C}$, which is the only one needed below, we can alternatively refer to [21, Corollary (a) to Theorem 1.14].) □

Proof of the theorem. Naturally, the claim C_1 depends only on the assumptions A_1 and A_2: For $z \in G$ fixed, the representation

$$\partial_z^{n+1} f(z,\omega) = \lim_{k\to\infty,\, z+k^{-1}\in G} k \cdot (\partial_z^n f(z+k^{-1},\omega) - \partial_z^n f(z,\omega)) \qquad (\omega \in \Omega,\ n \in \mathbf{N}_0)$$

shows that $\partial_z^n f(z,\cdot)$ is measurable for every $z \in G$ and every $n \in \mathbf{N}_0$. The product measurability of each function $\partial_z^n f$ follows using the preceding Lemma. Let now $A \subset G$. Choose a countable and dense subset D of A. Using the continuity of $\partial_z^n f(\cdot,\omega)$, the measurability of $\sup_{z\in A} |\partial_z^n f(z,\cdot)| = \sup_{z\in D} |\partial_z^n f(z,\cdot)|$ follows.

Without loss of generality, we may assume in what follows that μ is σ-finite. To prove this claim, let $\Omega_z := \{\omega \in \Omega : |f(z,\omega)| > 0\}$ for $z \in G$. Let further $D \subset G$ be countable and dense. By assumption A_3, we have $f(z,\cdot) \in L^1(\mu)$, so that Ω_z must be σ-finite for every $z \in G$. Using the continuity of $f(\cdot,\omega)$, we get $\tilde{\Omega} := \bigcup_{z\in G} \Omega_z = \bigcup_{z\in D} \Omega_z$ and hence the σ-finiteness of $\tilde{\Omega}$. Obviously, validity of the theorem for the given measure μ is implied by the validity of the theorem for the σ-finite measure $\tilde{\mu} := \mu(\cdot \cap \tilde{\Omega})$.

Proof of C_2. We use the standard notation $B_r(a) := \{z \in \mathbf{C} : |z-a| < r\}$ for open disks. Let $a \in G$ and $r \in]0,\infty[$ be such that the closed disk $\overline{B_r(a)} \subset G$. If g is any function holomorphic in G, the Cauchy formulas for the derivatives yield

$$g^{(n)}(z) = \frac{n!}{2\pi} \int_0^{2\pi} \frac{g(a+re^{it})}{(a+re^{it}-z)^{n+1}} re^{it}\, dt \qquad (z \in B_r(a)). \quad (10)$$

For $z \in B_{r/2}(a)$, we have $|a + re^{it} - z| \geq r/2$, and hence

$$|g^{(n)}(z)| \leq \frac{n!}{2\pi} \frac{2^{n+1}}{r^n} \int_0^{2\pi} |g(a+re^{it})|\, dt \qquad (z \in B_{r/2}(a)). \quad (11)$$

An application of (11) to each $f(\cdot,\omega)$ yields

$$\int \sup_{z\in B_{r/2}(a)} |\partial_z^n f(z,\omega)|\, d\mu(\omega) \leq \frac{n!2^{n+1}}{2\pi r^n} \int \int_0^{2\pi} |f(a+re^{it},\omega)|\, dt\, d\mu(\omega).$$

A change of the order of integration on the right hand side (which is allowed, since μ is assumed to be σ-finite and the integrand is nonnegative and known to be product measurable) yields, using assumption A_3, finiteness of the right hand side, and thus finiteness of the left hand side. This implies (2) via the usual covering argument.

First proof of C_3. For $n = 0$, the claim (3) is trivial. To proceed from n to $n+1$, thus proving in particular the claimed holomorphy, calculate for fixed $z \in G$ the difference quotients of $\int \partial_z^n f(\cdot,\omega)\, d\mu(\omega)$ under the integral. Applying the mean value theorem of differential calculus (in the inequality version valid for vector-valued functions [5, § 8.5.4]) in a compact and convex neighbourhood of z in G, and using C_2 with $n+1$ in place of n, we easily see that the dominated convergence theorem is applicable, yielding (3) with $n+1$ in place of n.

Second proof of C_3. To abbreviate, let $F := \int f(\cdot,\omega)\, d\mu(\omega)$. From claim C_2, already proved, for $n = 0$, we get the continuity of F using the dominated convergence theorem. We now use the characterization of holomorphy in terms of Cauchy's formula: Inserting F for g on the right hand side of (10) for $n = 0$ and interchanging the integrations shows that (10) holds for $n = 0$ with $g = F$. By [17 or 18, Theorem 8.2.1], F must be holomorphic. Finally an application of (10) for arbitrary n to $g = F$ yields, via interchanging integrations and an application of (10) to $g = f(\cdot,\omega)$, the formula (3). □

Proof of the addendum to the theorem. Modification of the proof concerning σ-finiteness: By (9), the set $G_0 := \{z \in G : \int |f(z,\omega)|\, d\mu(\omega) < \infty\}$ differs from the open set G at most by a set of two-dimensional Lebesgue measure zero, and hence is dense in G. Therefore, choosing $D \subset G_0$ countable and dense, D is also dense in G, so that we may proceed as in the proof of the theorem.

Modification of the proof of C_2. We now have to integrate over an area rather than over a curve. Here is the version of the argument suggested by the referee. Let $z \in G$ and $r \in]0,\infty[$ with $\overline{B_r(z)} \subset G$. For any g holomorphic in G, take (10) with $a = z$, write s in place of r, and apply $(n+2)r^{-(n+2)} \int_0^r \dots s^{n+1}\, ds$. The result is

$$g^{(n)}(z) = \frac{n+2}{r^{n+2}} \frac{n!}{2\pi} \int_0^r s \int_0^{2\pi} g(z+se^{it}) e^{-int}\, dt\, ds,$$

implying

$$|g^{(n)}(z)| \leq \frac{n+2}{r^{n+2}} \frac{n!}{2\pi} \iint_{B_r(z)} |g(\xi + i\eta)|\, d\xi d\eta,$$

and hence, for $z_0 \in G$ and $r \in]0,\infty[$ with $\overline{B_{2r}(z_0)} \subset G$,

$$\sup_{|z-z_0|\leq r} |g^{(n)}(z)| \leq \frac{n+2}{r^{n+2}} \frac{n!}{2\pi} \iint_{B_{2r}(z_0)} |g(\xi + i\eta)|\, d\xi d\eta.$$

With $f(\cdot,\omega)$ in place of g and with $2r \leq \delta$ with δ from (9), an application of $\int \dots d\mu(\omega)$ yields (2). □

Proof of the side remark. Put $h := \int |f(\cdot,\omega)|\, d\mu(\omega)$ and

$$G^* := \Big\{ z_0 \in G : \exists\, \delta > 0 \text{ with } \sup_{z\in G, |z-z_0|\leq\delta} h(z) < \infty \Big\}.$$

Then G^* is open. By Fatou's lemma, h is lower semicontinuous. An easy application of Baire's theorem [6, § 12.16.2] shows that G^* is dense in G. Obviously, the assumptions of the theorem, and hence its conclusions, hold with G^* in place of G. □

Construction of Counterexample 1. Let us fix $\nu \in \mathbf{N}$ for a moment and put

$$\begin{aligned} A_\nu &:= \{z \in \mathbf{C} : |z| \le 1/\nu\}, \\ B_\nu &:= \{z \in \mathbf{C} : |z| \le \nu,\ d(z, [0,\infty[) \ge 2/\nu\}, \\ C_\nu &:= \{z \in \mathbf{C} :\ d(z, [3/\nu, \nu]) \le 1/\nu\}. \end{aligned}$$

Then the sets A_ν, B_ν, C_ν are compact and disjoint. Setting $K_\nu := A_\nu \cup B_\nu \cup C_\nu$, we observe that $\mathbf{C} \setminus K_\nu$ is connected. Runge's theorem on polynomial approximation [21, Theorem 13.7] yields the existence of a polynomial q_ν with

$$\begin{aligned} |q_\nu(z)| &\le 2^{-\nu} \qquad (z \in B_\nu \cup C_\nu), \\ |q_\nu(z) - \sum_{n=0}^{\nu} a_n z^n/n!| &\le 2^{-\nu} \qquad (z \in A_\nu). \end{aligned}$$

We now define $p_0 := q_0$ and $p_\nu := q_\nu - q_{\nu-1}$ for $\nu \ge 2$. It is not difficult to check that the sequence $(p_\nu : \nu \in \mathbf{N})$ has the required properties. To derive appropriate bounds for nth derivatives, one may use (10) with $a = z$ and $r = 1/\nu$.

Construction of Counterexample 2. For $k \in \mathbf{N}$, let

$$A_k := \{\rho e^{i\theta} :\ \rho \in [1/k, k],\ \theta \in [1/k, 2\pi]\}.$$

Using Runge's theorem, we can easily find a polynomial r_k with

$$r_k(0) = 0, \quad r_k'(0) = 1, \quad |r_k(z)| \le \frac{1}{k^2} \qquad (z \in A_k).$$

Let us put $\lambda_k := \sup_{|z|\le k} |r_k(z)|$ and choose a sequence $(l_k : k \in \mathbf{N})$ of even positive integers satisfying $\lim_{k\to\infty} \lambda_k/l_k = 0$. We now write $q_k := l_k^{-1} r_k$ and define the polynomial p_ν to be the νth element in the sequence

$$(-)^0 q_1, (-)^1 q_1, \ldots, (-)^{l_1-1} q_1,\ (-)^0 q_2, (-)^1 q_2, \ldots, (-)^{l_2-1} q_2, \ldots$$

If now $z \in \mathbf{C}$ and $N \in \mathbf{N}$ with $\sum_{j=1}^{k-1} l_j < N \le \sum_{j=1}^{k} l_j$ and with $k \ge |z|$, then

$$|\sum_{\nu=1}^{N} p_\nu(z)| \in \{0, |q_k(z)|\} \le \frac{\lambda_k}{l_k}.$$

Hence we get (6), with the limit being identically zero. For $z = 0$, (7) is trivial. For $z \ne 0$, there is a $k_0 \in \mathbf{N}$ with $z \in A_k$ for $k \ge k_0$. Hence, with $\nu_0 := \sum_{j=1}^{k_0-1} l_j$,

$$\sum_{\nu > \nu_0} |p_\nu(z)| = \sum_{k \ge k_0} l_k |q_k(z)| = \sum_{k \ge k_0} |r_k(z)| \le \sum \frac{1}{k^2} < \infty,$$

yielding (7). Finally, (8) is obvious: $\sum_\nu |p_\nu'(0)| = \sum_k l_k q_k'(0) = \sum_k 1 = \infty$.

Acknowledgement
The present work was done while the author held a Heisenberg grant of the Deutsche Forschungsgemeinschaft, at Fachbereich Mathematik of Universität Hamburg. I thank R.B. Burckel, J. Elstrodt, W.K. Hayman, M. Reinders, R. Remmert, S. Saeki, and the referee for helpful remarks on earlier versions.

References

1 Averna, D. Sulla misurabilità delle funzioni di più variabili. *Rend. Circ. Mat. Palermo (2)* **35** (1986), 22–31.

2 Burckel, R.B. *An Introduction to Classical Complex Analysis, Vol. 1*, Birkhäuser, Basel, 1979.

3 Burckel, R.B. *Saekis Konstruktion.* (2 pp.), unpublished manuscript, 1991.

4 Chen, Y. and W.K. Hayman. On a non-regular parametric integral. III. *Complex Variables Theory Appl.* **37** (1998), 113–122.

5 Dieudonné, J. *Foundations of Modern Analysis.* Academic Press, New York, 1960.

6 Dieudonné, J. *Treatise on Analysis, Vol. II.* Academic Press, New York, 1970.

7 Dudley, R.M. *Real Analysis and Probability.* Wadsworth, Belmont, 1989.

8 Elstrodt, J. *Maß- und Integrationstheorie.* Springer, Berlin, 1996.

9 Everitt, W.N. and W.K. Hayman. On a non-regular parametric integral. II. *Complex analysis and differential equations* (Uppsala 1997), 146–156, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist. **64**, Uppsala Univ., Uppsala, 1999.

10 Everitt, W.N., W.K. Hayman, and G. Nasri-Roudsari. On the representation of holomorphic functions by integrals. *Appl. Anal.* **65** (1997), 95–102.

11 Hayman, W.K. On a non-regular parametric integral. *Proc. Roy. Soc. Edinburgh Sect. A* **83** (1979), 185–188.

12 Königsberger, K. *Analysis 2, 2nd ed.* Springer, Berlin, 1997..

13 Lebesgue, H. Sur l'approximation des fonctions. *Bull. Sci. Math.* **22** (1898), 10 pages. Reprinted in [14].

14 Lebesgue, H. *Oeuvres Scientifiques, Vol. III.* L'Ens. Mathématique, Genève, 1972.

15 Mattner, L. Product measurability, parameter integrals, and a Fubini counterexample. *Enseign. Math.* **45** (1999), 271–279.

16 Osgood, W.F. Note on the functions defined by infinite series whose terms are analytic functions of a complex variable; with corresponding theorems for definite integrals. *Ann. of Math. (2)* **3** (1901-02), 25–34.

17 Remmert, R. *Funktionentheorie 1, 4th ed.* Springer, Berlin, 1995.

18 Remmert, R. *Theory of Complex Functions.* Springer, New York, 1991. Translation of the second edition of [17].

19 Remmert, R. *Funktionentheorie 2, 2nd ed.* Springer, Berlin, 1995.

20 Remmert, R. *Classical Topics in Complex Function Theory.* Springer, New York, 1998. Translation of [19].

21 Rudin, W. *Real and Complex Analysis, 3rd ed.* McGraw-Hill, 1987.

22 Świątkowski, T. On the holomorphism of the integral with respect to a complex parameter. *Colloq. Math.* **16** (1967), 61–65.