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Oscillations of the Taylor polynomials for the sin function

The n-th order Taylor polynomial of $y = \sin x$ approximates the oscillations of this function, but it can have at most n zeros. In this note, it is shown that asymptotically for $n \to \infty$, it has about $\frac{2}{\pi e} n$ zeros.

Dedicated to my son Tilman

With the advent of graphing calculators, it has become a nice and rather easy activity to plot Taylor polynomials and to check how well or bad they approximate a given function. After having done that up to some degree, say $N \leq 10$, the more mathematically minded students wonder what happens in the limit of large N. The remainder formula of the Taylor series gives an upper bound for the error, which one expects to be of the right order of magnitude. In this note we get to more precise information about a specific example. We have chosen the function $y = \sin x$ with its infinitely many zeros. But its N-th MacLaurin polynomial P_N can have at most N zeros. How much of that "ability to oscillate" is actually occuring? We prove

$$\lim_{N \to \infty} \frac{c_N}{N} = \frac{2}{\pi e}, \tag{1}$$

where c_N is the number of real zeros of P_N, counting their multiplicities.

The easier part is the lower bound for the number of zeros. Using the remainder term of the Taylor series, this is done in Lemmas 1 through 3. Lemma 4 through 9 produce an upper bound. One has to rule out extra oscillations of the Taylor polynomials beyond the natural oscillations from the trigonometric function. To this end, we use an argument well known for the comparison of solutions of Sturm Liouville problems (see e.g. [1] p.208).

The $2n+1$-th order MacLaurin polynomial of the function $y = \sin x$ is

$$P_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \tag{2}$$

To avoid some complications arising from the alternating signs, we assume $N = 4n+1$. The case $N = 4n-1$ is of course exactly similar.

The lower bound

This is the easier part. We need Lemmas 1 through 3.

Lemma 1. *$P_{4n-1}(x) < \sin x < P_{4n+1}(x)$ for all $x > 0$ and $n \in \mathbf{N}$.*

Proof. This is straightforward to get by repeated integrations, starting with the estimate $\sin x < x$ for all $x > 0$. □

Lemma 2. *If for all $0 < x \leq (2m - \frac{1}{2})\pi$ and any $m \in \mathbf{N}$*

$$P_{4n+1}(x) - P_{4n-1}(x) \leq 1, \tag{3}$$

then

$$c_{4n+1} \geq 4m + 1. \tag{4}$$

Proof. Because of (1) from Lemma 1, the assumption (2) implies

$$0 < P_{4n+1}(x) - \sin x < 1 \tag{5}$$

for all $0 < x \leq (2m - \frac{1}{2})\pi$. At $x_k = (2k - \frac{3}{2})\pi$ and $z_k = (2k - \frac{1}{2})\pi$ with $k = 1, 2, \cdots, m$, the sin function takes its maximal and minimal values $+1$ and -1. Hence estimate (6) implies $P_{4n+1}(x_k) > 1$ and $P_{4n+1}(z_k) < 0$. By the intermediate value theorem, the polynomial $P_{4n+1}(x)$ has at least $2m-1$ zeros in the interval $(\frac{\pi}{2}, \frac{(2m-1)\pi}{2})$ and a further bigger zero occurs since $P_{4n+1}(x)$ tends to $+\infty$ for $x \to \infty$. Now the assertion follows since $P_{4n+1}(x)$ is odd. □

Lemma 3. *We have $c_{4n+1} \geq 1 + 4 \left\lfloor \frac{1}{4} + \frac{\sqrt[4n+1]{(4n+1)!}}{2\pi} \right\rfloor$.* (6)

Proof. For $0 < x \leq \sqrt[4n+1]{(4n+1)!}$, we estimate

$$P_{4n+1}(x) - P_{4n-1}(x) = \frac{x^{4n+1}}{(4n+1)!} \leq 1.$$

Assumption (4) of Lemma 2 holds with $m \in \mathbf{N}$ the largest integer such that $(2m - \frac{1}{2})\pi \leq \sqrt[4n+1]{(4n+1)!}$. Hence (5) implies the estimate (7) to be shown. □

Proof of the lower bound. To finish the proof, we still need

$$\lim_{N \to \infty} \frac{\sqrt[N]{N!}}{N} = \frac{1}{e}, \tag{7}$$

which is an easy consequence of Stirling's formula. Alternatively, one can get (8) from Polya and G. Szego [2], part I chapter 1.1, number 69. In the limit $n \to \infty$, the lower estimate (7) implies

$$\liminf_{n \to \infty} \frac{c_{4n+1}}{4n+1} \geq \liminf_{n\to\infty} \frac{2 \sqrt[4n+1]{(4n+1)!}}{\pi(4n+1)} = \frac{2}{\pi e}. \qquad \square$$

The upper bound

It needs a bit more work to get an upper estimate for the number of zeros. In Lemma 4 through 6, we show that the Taylor polynomial has at most two zeros per period. In Lemmas 7 through 9, we get a sharp upper estimate for the largest zero of the Taylor polynomial.

Lemma 4. *All zeros of the polynomial P_{4n+1} have at most multiplicity two. If $x > 0$ and $P_{4n+1}(x) = P'_{4n+1}(x) = 0$, then $P_{4n+1}(z) > 0$ for $z \neq x$ and $|z - x|$ small enough.*

Proof. Let $x > 0$ be a double zero of P_{4n+1}. Because of

$$P''_{4n+1}(x) + P_{4n+1}(x) = \frac{x^{4n+1}}{(4n+1)!} > 0,$$

the multiplicity of the zero cannot be higher than two and P_{4n+1} assumes a local minimum at x. $\square$

Lemma 5. *If $0 < a < b$ are two successive zeros of P_{4n+1} and $P_{4n+1}(x) > 0$ for all $x \in (a,b)$, then $b - a > \pi$.*

Proof. We use Green's formula

$$\int_a^b [(f'' + f)\phi - f(\phi'' + \phi)]\, dx = \left[f'\phi - f\phi'\right]_a^b$$

for the functions $f = P_{4n+1}$, $\phi = \sin \frac{\pi(x-a)}{b-a}$ and get

$$\int_a^b \left[\frac{x^{4n+1}}{(4n+1)!} - P_{4n+1}(x) \left(1 - \frac{\pi^2}{(b-a)^2}\right)\right] \phi(x)\, dx = 0.$$

Since $\phi(x) > 0$ and $P_{4n+1}(x) > 0$ for $a < x < b$, we conclude that $b - a > \pi$. $\square$

Lemma 6. *Let X_N be the largest zero of the Taylor polynomial $P_N(x)$. Then we have*

$$c_{4n+1} \leq 1 + 4\left\lceil \frac{X_{4n+1}}{2\pi} \right\rceil. \tag{8}$$

Proof. The ceiling term is the minimal natural number m such that $2\pi m > X_{4n+1}$. There are at most two zeros of P_{4n+1} in each of the m intervals $(\pi, 2\pi), (3\pi, 4\pi), \ldots$ up to $((2m-1)\pi, 2m\pi)$, which include all positive zeros. Hence, by symmetry, this polynomial has at most $1 + 4m$ real zeros. $\square$

It remains to get a precise upper estimate of X_{4n+1}. Lemma 7 the first natural attempt. The reader should convince himself that it is not strong enough to get the final result, but Lemma 9 indeed is.

Lemma 7. *If $x > 0$ and $x^2 \geq 4n(4n+1)$, then $P_{4n+1}(x) > x > 0$.*

Proof. We group the terms of the polynomial P_{4n+1} to positive pairs to get

$$P_{4n+1}(x) = x + \sum_{k=1}^{n} \frac{x^{4k-1}}{(4k-1)!}\left[\frac{x^2}{4k(4k+1)} - 1\right] > x > 0. \qquad \square$$

Lemma 8. *If $x > 0$ and $P_{4n+1}(x) - P_{4n+5}(x) > 1$, then $P_{4n+1}(x) > 0$.*

Proof. Lemma 1 implies $\sin x < P_{4n+5}(x) < P_{4n+1}(x) - 1$ and hence $P_{4n+1}(x) > 1 + \sin x \geq 0$. $\square$

Lemma 9. *If $\frac{4(8n+5)\, x^{4n+3}}{(4n+5)!} \geq 1$, then $P_{4n+1}(x) > 0$.*

Proof. We distinguish the cases (i) $x^2 \geq 4n(4n+1)$ and (ii) $x^2 < 4n(4n+1)$. In the first case, the result follows from Lemma 7. In the second case, we estimate

$$\begin{aligned} P_{4n+1}(x) - P_{4n+5}(x) &= \frac{x^{4n+3}}{(4n+3)!}\left[1 - \frac{x^2}{(4n+4)(4n+5)}\right] \\ &> \frac{x^{4n+3}}{(4n+3)!}\left[1 - \frac{4n(4n+1)}{(4n+4)(4n+5)}\right] \\ &= \frac{x^{4n+3}\, 4(8n+5)}{(4n+5)!} \geq 1, \end{aligned}$$

and use Lemma 8. $\square$

Proof of the upper estimate. Lemma 9 implies

$$X_{4n+1} < \sqrt[4n+3]{\frac{(4n+5)!}{4(8n+5)}},$$

and using (10), and once more Stirling's formula or (8), we get in the limit $n \to \infty$

$$\begin{aligned} \limsup_{n \to \infty} \frac{c_{4n+1}}{4n+1} &\leq \lim_{n\to\infty} \frac{2 \sqrt[4n+3]{(4n+3)!}}{\pi(4n+1)} \sqrt[4n+3]{\frac{(4n+4)(4n+5)}{4(8n+5)}} \\ &= \frac{2}{\pi e}. \end{aligned} \qquad \square$$

References

1 Earl A. Coddington and Norman Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill 1955.

2 G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Volume I, Springer New York Heidelberg Berlin, 1972.