Problemen

| Problem Section

Problem 7 (H. van den Berg)
For a natural number n not divisible by 3 show that the only integer solutions of

$$(X^2 - YZ)^n + (Y^2 - XZ)^n + (Z^2 - XY)^n = 1$$

are the trivial ones.

Problem 8 (Ronald de Man)
Partition the set of numbers $\{1, 2, \ldots, 2n\}$ into n pairs and denote the set of pairs by A. Define a partial order on A by $\{a_i, b_i\} < \{a_j, b_j\}$ if and only if $a_j < a_i < b_i < b_j$. As usual, we call a totally ordered subset of $(A, <)$ a chain and we say that two chains A_1, A_2 cover A if $A_1 \cup A_2 = A$. Count the number of partitions into pairs for which A can be covered by two chains.

Problem 9 (Alex Heinis)
Let a trapezoid with a right angle be given. Suppose that there is a circle that is tangent to all four sides of the trapezoid. Let L be the line through the intersection of the diagonals, parallel to the base of the trapezoid. Show that the length of the segment on L bounded by two sides is equal to the height of the trapezoid.

Solutions to volume 1, no 1 (March 2000)

Problem 1 A solid cylinder is placed on a horizontal plane. Suppose that the barycentre is not on the central axis, nor vertically below this axis, so that gravity forces the cylinder into oscillation. Moreover, suppose that the oscillations are infinitesimal and that rolling is frictionless. Then what is the period of the oscillation?

The original solution by H.A. Lorentz appeared in the first issue of Nieuw Archief in 1875. In fact, Lorentz gave two solutions to the problem, interpreting the frictionless rolling of the cylinder in two ways: rolling on ice or rolling on a table without energy loss. The solution below, due to Beukers, deals with both cases in one go.

Solutions by Frits Beukers (Utrecht) and Fons Daalderop (Delft).

In the cylinder, which we assume to be in stable rest, we take cylindrical coordinates z, r, ϕ. The direction $\phi = 0$ corresponds to the direction from the central axis to the area that touches the table on which the cylinder is resting. By $\int f\, dm$ we denote integration of the quantity f over the whole cylinder with respect to the mass distribution. The total mass equals $m = \int \rho\, dV$, the moment of inertia is given by $I = \int r^2\, dm$. Due to the fact that the cylinder is stably at rest we have $\int r \sin\phi\, dm = 0$ and the distance of the center of gravity to the central axis is given by $r_z = \int r \cos\phi\, dm$.

Let r_0 be the radius of the cylinder. If we slightly move the cylinder, it undergoes a rotation by some small angle we call θ, and a horizontal movement which is about $r_0\theta$. The point with cylindrical coordinates z, r, ϕ undergoes a displacement by the vector

$$(r_0\theta, 0) + r(\sin\phi, -\cos\phi)\theta.$$

Note that if the cylinder is rolling on ice instead of on a table, the term $(r_0\theta, 0)$ vanishes. The kinetic energy corresponding to this displacement is

$$E_{\text{kin}} = \frac{1}{2}\left(\int ((r_0 + r\sin\phi)^2 + r^2\cos^2\phi) dm\right)\dot{\theta}^2,$$

where $\dot{\theta}$ denotes the derivative of θ with respect to time. We derive

$$E_{\text{kin}} = \frac{1}{2}\left(\int (r_0^2 + r^2 + 2rr_0\sin\phi) dm\right)\dot{\theta}^2 = \frac{1}{2}(mr_0^2 + I)\dot{\theta}^2.$$

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

Solutions to problems from the fourth series will be published in the next issue.

Editor:
R.J. Fokkink
Technische Universiteit Delft
Faculteit Wiskunde
P.O. Box 5036
2600 GA Delft
The Netherlands
r.j.fokkink@its.tudelft.nl

Oplossingen | Solutions

The potential energy of the displaced cylinder is given by $E_{\text{pot}} = -\int gr\cos(\phi-\theta)\,dm$, where g is the acceleration of gravity. Using the approximation $\cos(\phi-\theta) = \cos\phi + (\sin\phi)\theta - \frac{1}{2}(\cos\phi)\theta^2 + O(\theta^3)$, we get $E_{\text{pot}} = -mgr_z + \frac{1}{2}mgr_z\theta^2$. The total energy, forgetting about the constant term $-mgr_z$, now equals

$$\frac{1}{2}\left((mr_0^2+I)\dot{\theta}^2 + mgr_z\theta^2\right).$$

Note that this is precisely the energy function of a harmonic oscillator with angular frequency $\sqrt{\frac{mgr_z}{mr_0^2+I}}$. If the cylinder is rolling on ice, the term mr_0^2 in the denominator vanishes.

Problem 2 One quickly verifies that the plane cannot be partitioned in two sets, such that points of distance 1 are in opposite sets. A point in the plane is called *rational* if both its coordinates are rational. Is it possible to partition the rational points in two sets, such that points of distance 1 are in opposite sets?

In the spring of 1999, Wim Feijen (Eindhoven) put the following teaser on a blackboard in a hallway of the computer science department at TU Eindhoven: "Prove that there is no coloring by three colors of the real plane such that every pair of points of distance 1 *is of a different color." Having solved this problem, a natural question is how many colors are needed for the rational plane.*

This problem was brought to CWI Amsterdam by Sjouke Mauw. Several people in Eindhoven and Amsterdam solved the problem. A neat solution is on Vincent van Oostrom's homepage at http://www.phil.uu.nl/~oostrom/publication/misc.html. His solution is similar to the one below. It is still unknown how many colors are required for the real plane. This is known as Hadwiger's Problem.

Solutions by Ronald van Luijk (Utrecht), Frits Beukers (Utrecht) and Robert Israel (Vancouver). Their solutions are the same.

The answer is 'yes'. Color the points in $\mathbf{Q}^2$ red and blue as follows.

Consider the set $V = \{(\frac{a}{2^k}, \frac{b}{2^k}) \mid a,b,k \in \mathbf{Z}_{\geq 0},\ 0 \leq a,b < 2^k\}$. We color all points of V red. Note that any point $(p,q) \in \mathbf{Q}^2$ can be written uniquely in the form $(p,q) = (p',q') + (\alpha,\beta)$ where $(\alpha,\beta) \in V$ and p', q' have odd denominators. We color (p,q) red if $p'+q' \equiv 0 \pmod 2$ and blue if $p'+q' \equiv 1 \pmod 2$. This gives the desired partitioning.

To see this, observe that $r^2+s^2 = 1$ with $r,s \in \mathbf{Q}$ is only possible if r,s have odd denominators. In particular, $r+s \equiv r^2+s^2 \equiv 1 \pmod 2$. Hence $(p_1-p_2)^2 + (q_1-q_2)^2 = 1$ with $p_i, q_j \in \mathbf{Q}$ implies that p_1-p_2, q_1-q_2 have odd denominators. Moreover, $p_1-q_1+p_2-q_2 \equiv 1 \pmod 2$ from which we infer $p_1+q_1 \equiv p_2+q_2+1 \pmod 2$.

Problem 3 The sequence $a_1, a_2, \ldots$ is defined by $\sum_{d|n} a_d = 2^n$ for all $n \geq 1$. Show that $n|a_n$ for all n.

Lucien van Hamme (Brussel) points out that this is a classical result, going back to 1880 or earlier, and refers to Dickson's History of the theory of numbers*. Van Hamme remarks that the problem can be generalized considerably, as shown by C. Smyth,* A coloring proof of a generalization of Fermat's little theorem*, Amer. Math. Monthly* **93** *(1986), 469–471. Some readers may have noticed that Problem 3 is in fact equivalent to Problem 977 in Nieuw Archief, vierde serie, no. 15 (3) (1997).*

Solutions by Ronald van Luijk (Utrecht), Frits Beukers (Utrecht), Jack van Lint (Pasadena), Wim Luxemburg (Pasadena), Minh Can (Columbus), Robert Israel (Vancouver). Below is the solution by Van Lint and Luxemburg. It is a pretty little counting argument, submitted on the same day, from the same place, without the one being aware of the other's solution.

Let $M(d)$ denote the number of circular sequences of 0's and 1's of length d that are *not* periodic. Counting *all* circular sequences of 0's and 1's, we find $\sum_{d|n} dM(d) = 2^n$. Since $a_n = nM(n)$ by definition, we are done.