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Minimization of the renormalized energy in the unit ball of $\mathbf{R}^2$

We establish an explicit formula for the renormalized energy corresponding to the Ginzburg-Landau functional. Then we find the location of vortices in the case of the unit ball in $\mathbf{R}^2$, provided that the topological Brouwer degree of the boundary data equals to 2 or 3. Our proofs use techniques related to linear partial differential equations (Green's formula for the Neumann problem), convex functions, elementary identities or inequalities in the complex plane.

Superconductivity was discovered in 1911 by the Dutch physicist Kamerlingh-Onnes. Superconducting materials exhibit two main properties:
i. Their electric resistance is virtually zero.
ii. They have peculiar magnetic behavior.
From this point of view, superconductors can be classified into two types. In type I, magnetic fields are excluded from the material (except for a very thin layer near the surface). Type II superconductors, on the other hand, do allow penetration of magnetic fields, but these fields concentrate in narrow regions or points, called *vortices*. In fact, type II superconductors can sustain very high magnetic fields.

The first successful theory for superconductivity was the phenomenological macroscopic model proposed in 1935 by London. His theory accounted for the expulsion of magnetic fields and predicted the quantization of magnetic fluxoids. Then, in 1950 Ginzburg and Landau [3] proposed a more involved theory which allowed for spatial variations of both the magnetic field and the superconductivity order parameter. In addition to the model's success in explaining the experimental observations of the day, it was by Abrikosov in 1957 to predict in [1] the existence of type II superconductors, and the formation of large array of magnetic vortices for such materials. In 1994, Bethuel, Brezis and Hélein proposed a mathematical model of the Ginzburg-Landau theory which relates the number of vortices to a topological invariant of the boundary condition. A fundamental role in their analysis is played by the notion of *renormalized energy*.

We give in what follows a partial answer to a problem raised by Bethuel, Brezis and Hélein in [2]. Let $B_1 = \{x = (x_1, x_2) \in \mathbf{R}^2;\ x_1^2 + x_2^2 = |x|^2 < 1\}$. Fix d a positive integer and consider a configuration $a = (a_1, \dots, a_d)$ of distinct points in B_1. Let $\rho > 0$ be sufficiently small such that the balls $B(a_i, \rho)$ are mutually disjoint and contained in B_1 and set $\Omega_\rho = B_1 \setminus \bigcup_{i=1}^d \overline{B(a_i,\rho)}$. Consider the boundary data $g : S^1 \to S^1$ defined by $g(\theta) = e^{id\theta}$. We observe that the Brouwer degree $\deg(g, S^1)$ is equal to d. We recall that if $G \subset \mathbf{R}^2$ is a smooth, bounded and simply connected domain and $h = (h_1, h_2) \in C^1(\partial G, S^1)$ then the topological Brouwer degree (i.e., the winding number of h considered as a map from ∂G into S^1) is defined by

$$\deg(h, \partial G) = \frac{1}{2\pi} \int_{\partial G} \left(h_1 \frac{\partial h_2}{\partial \tau} - h_2 \frac{\partial h_1}{\partial \tau} \right),$$

where τ denotes the unit tangent vector to ∂G.

In [2], F. Bethuel, H. Brezis and F. Hélein have studied the behavior as $\rho \to 0$ of solutions of the minimization problem

$$E_{\rho,g} = \min_{v \in \mathcal{E}_{\rho,g}} \int_{\Omega_\rho} |\nabla v|^2, \tag{1}$$

where

$$\mathcal{E}_{\rho,g} = \{v \in H^1(\Omega_\rho; S^1);\ v = g \text{ on } \partial G \text{ and } \deg(v, \partial B(a_i, \rho)) = +1, \text{ for } i = 1, ..., d\}.$$

We have denoted by $H^1(\Omega_\rho; S^1)$ the space of all measurable functions $u : \Omega_\rho \to \mathbf{R}^2$ such that $u \in H^1(\Omega_\rho)$ and $|u| = 1$ for a.e. $x \in \Omega_\rho$. We also point out that all the derivatives appearing in this paper are taken in distributional sense.

It is proved in [2] that problem (1) has a unique solution, say u_ρ. By analyzing the behavior of u_ρ as $\rho \to 0$ the following asymptotic estimate is obtained as well:

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla u_\rho|^2 = \pi d \log \frac{1}{\rho} + W(a) + O(\rho), \quad \text{as } \rho \to 0. \tag{2}$$

In [2], the functional $W(a)$ is implicitly defined by the formula

$$W(a) = -\pi \sum_{i\neq j} \log|a_i - a_j| + \frac{d}{2}\int_{S^1} \Phi d\sigma - \pi \sum_{i=1}^{d} R(a_i)\,, \quad (3)$$

where Φ is the unique solution of the linear Neumann problem

$$\begin{cases} \Delta\Phi = 2\pi \sum_{i=1}^{d} \delta_{a_i} & \text{in } B_1\,, \\ \frac{\partial\Phi}{\partial\nu} = d & \text{on } S^1\,, \\ \int_{S^1} \Phi = 0\,, \end{cases} \quad (4)$$

where ν is the outward normal to S^1 and δ_b denotes the Dirac mass concentrated at the point $b \in B_1$, and where $R(x) = \Phi(x) - \sum_{i=1}^{d} \log|x - a_i|$. We observe that R is a harmonic function in B_1, so $R \in C(B_1)$, which means that $R(a_i)$ makes sense. The functional W, called the *renormalized energy*, has the following interesting properties:

i. $W(a) \to +\infty$ as two of the points a_i coalesce;
ii. $W(a) \to +\infty$ as one of the points a_i tends to ∂B_1.

The asymptotic expansion (2) shows that the renormalized energy W is what remains in the energy after the singular *core energy* $\pi\, d \log \frac{1}{\rho}$ has been removed.

The renormalized energy may be also obtained by changing the class of testing functions and adding a penalization in the energy. Such a penalty is $\frac{1}{\varepsilon^2}\int_{B_1}(1-|u|^2)^2$ which leds naturally to the Ginzburg-Landau functional

$$E_\varepsilon(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B_1} (1-|u|^2)^2\,, \qquad \varepsilon > 0\,.$$

Set $H^1_g = \{u \in H^1(B_1;\mathbf{C});\, u = g \text{ on } S^1\}$. As proved in [2] the minimization problem

$$\inf_{u\, \in H^1_g} E_\varepsilon(u)$$

has at least one smooth solution u_ε. Moreover u_ε converges (as $\varepsilon \to 0$) to a map with values in S^1 and which is C^∞, except for some configuration of points, called *vortices*. It is very surprising that this configuration consists exactly of d points. This shows that the topological degree of the boundary condition creates the same *quantized vortices* as a magnetic field in type II superconductors or as an angular rotation in superfluids (see [2], p. xviii). In [2] it is also proved that the configuration of d vortices is a global minimum point of the renormalized energy $W(a)$ with respect to all configurations of d distinct points in B_1. So the renormalized energy plays a crucial role in order to locate the singularities. The asymptotic expansion in this case (see [2], Chapter IX) is

$$E_\varepsilon(u_\varepsilon) = \pi\, d\, \log\frac{1}{\varepsilon} + \min_a W(a) + d\gamma + o(1) \qquad \text{as } \varepsilon \to 0\,,$$

where γ is some universal constant.

In [2], Chapter XI, Open Problem 12, it is asked whether the vortices form a regular configuration. The aim of this paper is to deduce with elementary arguments an *explicit* formula for the renormalized energy defined in (3) which will enable us to answer partially the question raised by Bethuel, Brezis and Hélein in their book. More precisely we prove

Theorem. *The expression of the renormalized energy is given by*

$$W(a) = -\pi \sum_{1\le i<j\le d} \log \mid a_i - a_j \mid^2 - \pi \sum_{i,j=1}^{d} \log \mid 1 - a_i \bar{a}_j \mid\,. \quad (5)$$

If $d = 2$ then the minimal configuration for W is unique (up to a rotation) and consists of two points which are symmetric with respect to the origin. If $d = 3$ then the configuration which minimizes W is also unique and it consists of an equilateral triangle centered at the origin.

Proof. We shall use the expression (3) for the renormalized energy $W(a)$. We observe that it suffices to compute the function R for one point, say a.
For every $a \neq 0$, let $a^\star = \frac{a}{|a|^2}$. We define the function $\mathcal{G} : B_1 \setminus \{a\} \to \mathbf{R}$ by

$$\mathcal{G}(x) = \begin{cases} \frac{1}{2\pi}\log\mid x-a\mid + \frac{1}{2\pi}\log\mid x - a^\star\mid - \frac{1}{4\pi}\mid x\mid^2 + \mathcal{C} \\ \qquad\qquad \text{if } a \neq 0 \\ \frac{1}{2\pi}\log\mid x\mid - \frac{1}{4\pi}\mid x\mid^2 + \mathcal{C} \quad \text{if } a = 0 \end{cases}$$

and we choose the constant $\mathcal{C}$ such that

$$\int_{S^1} \mathcal{G} = 0.$$

It follows that, for every $a \in B_1$,

$$\mathcal{C} = \frac{1}{4\pi} + \frac{1}{2\pi}\log\mid a\mid, \quad (6)$$

if $a \neq 0$, and $C = \frac{1}{4\pi}$ if $a = 0$. The function $\mathcal{G}$ satisfies

$$\begin{cases} \Delta\mathcal{G} = \delta_a - \frac{1}{\pi} & \text{in } B_1 \\ \frac{\partial\mathcal{G}}{\partial\nu} = 0 & \text{on } \partial B_1 \\ \int_{\partial B_1} \mathcal{G} = 0\,. \end{cases} \quad (7)$$

It follows now from (4) that

$$\begin{cases} \Delta\left(\frac{\Phi}{2\pi}\right) = \delta_a & \text{in } B_1 \\ \frac{\partial}{\partial\nu}\left(\frac{\Phi}{2\pi}\right) = \frac{1}{2\pi} & \text{on } \partial B_1 \\ \int_{\partial B_1} \frac{\Phi}{2\pi} = 0\,. \end{cases}$$

Thus the function $\Psi = \frac{\Phi}{2\pi} - \frac{1}{4\pi}(\mid x\mid^2 - 1)$ satisfies

$$\begin{cases} \Delta\Psi = \delta_a - \frac{1}{\pi} & \text{in } B_1 \\ \frac{\partial\Psi}{\partial\nu} = 0 & \text{on } S^1 \\ \int_{S^1} \Psi = 0\,. \end{cases} \quad (8)$$

By uniqueness, it follows from (7) and (8) that

$$\frac{\Phi}{2\pi} - \frac{1}{4\pi}(\mid x\mid^2 - 1) = \frac{1}{2\pi}\log\mid x - a\mid + \frac{1}{2\pi}\log\mid x - a^\star\mid - \frac{1}{4\pi}\mid x\mid^2 + \mathcal{C}\,.$$

Taking into account the expression of $\mathcal{C}$ given in (6), as well as the link between Φ and R we obtain (5).

Let a and b be two distinct points in B_1. Then

$$-\frac{W}{\pi} = \log(|a|^2 + |b|^2 - 2|a|\cdot|b|\cdot\cos\varphi) + \log(1+|a|^2|b|^2 - 2|a|\cdot|b|\cdot\cos\varphi) + \log(1-|a|^2) + \log(1-|b|^2)\,,$$

where φ denotes the angle between the vectors $\overrightarrow{Oa}$ and $\overrightarrow{Ob}$. So, a necessary condition for the minimum of W is $\cos\varphi = -1$, that is the points a, O and b are colinear, with O between a and b. Hence one may suppose that the points a and b lie on the real axis and $-1 < b < 0 < a < 1$. Denote

$$f(a,b) = 2\log(a-b) + 2\log(1-ab) + \log(1-a^2) + \log(1-b^2)\,.$$

Since the function $\log(1-x^2)$ is concave on $(0,+\infty)$ it follows that

$$\log(1-a^2) + \log(1-b^2) \le 2\log\left(1 - \left(\frac{a-b}{2}\right)^2\right).$$

On the other hand, it is obvious that $1 - ab \le 1 + \left(\frac{a-b}{2}\right)^2$. Hence

$$f(a,b) \le f\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$$

which means that the maximum of f is achieved provided that $a = -b$. A straightforward calculation shows that $\max f = f(5^{-1/4}, -5^{-1/4})$, so $\min W = -\pi f(5^{-1/4}, -5^{-1/4})$.

For $d = 3$, in order to minimize the functional W given by (5), it is enough to maximize the functional

$$F(a) = \prod_{1\le i<j\le 3} |a_i - a_j|^2 \left(|a_i - a_j|^2 + (1-r_i^2)(1-r_j^2)\right) \cdot \prod_{i=1}^{3}(1-r_i^2)\,,$$

where $r_i = |a_i|$.
Using the elementary identity

$$3\sum_{i=1}^{3}|a_i|^2 = |\sum_{i=1}^{3} a_i|^2 + \sum_{1\le i<j\le 3}|a_i - a_j|^2$$

we find

$$3\sum_{i=1}^{3}|a_i|^2 \ge \sum_{1\le i<j\le 3}|a_i - a_j|^2\,. \qquad (9)$$

Put $S = \sum_{i=1}^{3} r_i^2$. We try to minimize F keeping S constant. Using (9), we have

$$\prod_{1\le i<j\le 3}|a_i - a_j|^2 \le \left(\frac{\sum_{1\le i<j\le 3}|a_i - a_j|^2}{3}\right)^3 \le (\sum_{i=1}^{3}|a_i|^2)^3 = S^3\,, \qquad (10)$$

$$\prod_{i=1}^{3}(1-r_i^2) \le \left(\frac{3-S}{3}\right)^3 \qquad (11)$$

and

$$\prod_{1\le i<j\le 3}\left(|a_i - a_j|^2 + (1-r_i^2)(1-r_j^2)\right) \le \left(\frac{\sum_{1\le i<j\le 3}(|a_i - a_j|^2 + (1-r_i^2)(1-r_j^2))}{3}\right)^3 \le \left(\frac{\sum 1 - \sum r_i^2 - \sum r_j^2 + \sum r_i^2 r_j^2 + \sum|a_i - a_j|^2}{3}\right)^3 \le \left(\frac{3 - 2S + \frac{S^2}{3} + 3S}{3}\right)^3 = \left(\frac{S^2 + 3S + 9}{3^2}\right)^3. \qquad (12)$$

We have applied here the elementary inequality

$$\sum_{1\le i<j\le 3} r_i^2 r_j^2 \le \frac{1}{3}\left(\sum_{i=1}^{3} r_i^2\right)^2.$$

From (10), (11) and (12) we find

$$F \le S^3 \cdot \left(\frac{3-S}{3}\right)^3 \cdot \left(\frac{S^2+3S+9}{3^2}\right)^3 = \frac{1}{3^9}(-S^4 + 27S)^3\,.$$

It follows that the maximum of F is achieved if $S = 3\cdot 4^{-1/3}$ and $\max F = 3^6 \cdot 4^{-4}$, with equality when we have equality in (10), (11) and (12), i.e., if and only if $a_2 = \varepsilon a_1, a_3 = \varepsilon^2 a_1$, where $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. This implies that $\min W = \pi \log\frac{2^8}{3^6}$. □

Open problems

We conclude this paper with the following open problems which were raised by Professor Haim Brezis:

1. Find the configuration which minimizes W given by (5), provided that $d \ge 4$. Is this configuration given by a regular d-gon (as for $d = 2, 3$) or does it consist of an Abrikosov lattice as $d \to +\infty$, as predicted in [2], p. 139?
2. Prove that the minimal configuration 'goes to the boundary', as $d \to \infty$, in the following sense: for given d, let $a = (a_1, \dots, a_d)$ be an arbitrary configuration which minimizes W and set $x_d = \min\{|a_i|; 1 \le i \le d\}$. Prove that $\lim_{d\to\infty} x_d = 1$.

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