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Spreading gossip efficiently

We consider the situation in which n people each know a secret, and by means of a series of bilateral conversations (regular telephone conversations, say) want to exchange all secrets. In such a conversation, the participants share all secrets that they know at the time.

Claim. At least $2n-4$ conversations are needed before everyone knows every secret.

Remark. For $n \geq 4$, $2n-4$ conversations suffice. For four persons A, B, C and D, say, take conversations AB, and CD, followed by AC and BD. For every additional person P, schedule one conversation AP, before A, B, C and D interchange their knowledge, and another conversation AP afterwards. For $n = 1, 2$, and 3, respectively, 0, 1, and 3 conversations are necessary and sufficient. Below, we give a proof of the claim based on induction on the number n of gossipers.

Background

This problem has been solved before by many others, see [1–3, 5–6], each proof having its own characteristics. They are all different, but most of them use a lemma expressing a strong property of a minimum size gossip network.

The concept of exchanging information over a network has been widely studied and besides *gossiping*, where everyone has a piece of information to be spread among all others, there is the notion of *broadcasting*, in which one piece of information, known to a single individual, has to be spread. A survey with 135 references is found in [8].

Additional features worth mentioning here are bounds on the number of rounds of gossips needed to spread all information. Here a *round* is a set of simultaneous telephone calls. In [4] it is proved that at least $\lceil \log_2 n \rceil$ rounds are needed for n even, and at least $\lceil \log_2 n \rceil + 1$ for n odd. Sharp results are described by [11]. Other related results are found in [7, 9–10].

Notation and definitions

Here we first introduce some notation and definitions used in the proof.

Let G_n denote a minimum length sequence of conversations in which n gossipers exchange their information. Let $\phi(n)$ denote the length of G_n. Each conversation can be labeled $(\{a, b\}, t)$, denoting a conversation between participants a and b at time t. Since we may assume that all values t are distinct, we may as well refer

to G_n as a sequence $(\{a_j, b_j\})_j$.
Such a sequence defines a partial order between the conversations, where $\{a_j, b_j\} \prec \{a_k, b_k\}$, if and only if $j < k$ and $\{a_j, b_j\} \cap \{a_k, b_k\} \neq \emptyset$. We say that $\{a_j, b_j\}$ precedes $\{a_k, b_k\}$, and $\{a_k, b_k\}$ succeeds $\{a_j, b_j\}$. If $\{a_k, b_k\}$ is the first successor of $\{a_j, b_j\}$ containing a_j, it is called its a_j-successor. The a_j-successor and the b_j-successor of $\{a_j, b_j\}$ are called its *direct successors*. Similarly we define the direct predecessors of a conversation. A conversation has at most two direct successors, and at most two direct predecessors. If a conversation $\{a, b\}$ does not have an a-successor this is a's *final* conversation; if $\{a, b\}$ does not have an a-predecessor, it is a's *first* conversation.
If there exists a sequence of direct successors from $\{a, b\}$ to $\{c, d\}$: $p_1 = \{a, b\}, p_2, \ldots, p_k = \{c, d\}$, where p_{j+1} is a direct successor of p_j, for each j, information flows from $\{a, b\}$ to $\{c, d\}$. We say that $\{a, b\}$ *reaches* $\{c, d\}$, and denote the existence of such a sequence by $\{a, b\} \rightsquigarrow \{c, d\}$.
Note that in a sequence of conversations as described above all secrets are exchanged, if and only if, for each pair of a first conversation $\{a, b\}$ and a final conversation $\{c, d\}$, $\{a, b\} \rightsquigarrow \{c, d\}$.
We will often use the observation that if G_n is a sequence of conversations in which all secrets are exchanged, then so is the sequence obtained from G_n by reversal of time. Let $\overleftarrow{G}_n$ denote this reversal of G_n.

Proof

Basis of induction. The claim is evidently true for $n \leq 2$, since the number of necessary conversations $\phi(n)$ is at least $0 \geq 2n - 4$, for $n \leq 2$. For $n > 2$ we distinguish between a number of cases, depending on the number of conversations a gossiper participates in.

Case 1. There is only one conversation with participant a, for some a. Let $\{a, X\}$ denote this conversation. After this conversation at least another $n - 2$ conversations are necessary to spread a's secret to $\{1, \ldots, n\} \setminus \{a, X\}$, since with each additional conversation the set of participants knowing a's secret grows by at most one. Similarly, at least $n - 2$ conversations precede $\{a, X\}$ in order to collect all secrets from $\{1, \ldots, n\} \setminus \{a, X\}$ at person X. In total at least $1 + 2(n - 2) = 2n - 3$ conversations are needed.

Case 2. There are two or more conversations between participants a and b. Delete from G_n all conversations between a and b (at least two), and replace in the remaining conversations a by b. The result is a sequence of conversations in which all secrets $\{1, \ldots, n\} \setminus \{a\}$ are exchanged. By induction this remaining set consists of at least $\phi_{n-1} \geq 2n - 6$ conversations. Hence G_n must contain at least $2 + 2n - 6 = 2n - 4$ conversations.

Case 3. There is a conversation $\{a, X\}$ where X already knows all gossips. Consider the last occasion of this kind. Then it must be a's final conversation. Let $\{a, b\}$ be a's first conversation. Assuming that cases 1 and 2 do not apply, we find that $b \neq X$. Delete conversations $\{a, b\}$ and $\{a, X\}$, and replace in the remaining conversations a by b. The result is again a sequence of conversations in which $n - 1$ secrets are exchanged. We find that $|G_n| \geq 2 + \phi_{n-1} \geq 2n - 4$.

Note that if none of cases above apply, we are in the situation in which each participant makes at least two conversations; two participants carry at most one conversation with one another; and in addition, if $\{a, b\}$ is a's final conversation, then this must be b's final conversation as well. Applying the observation on the reversed sequence $\overleftarrow{G}_n$ we also see that, if $\{a, b\}$ is a's first conversation, then this must be b's first conversation as well.

Case 4. There are only two conversations with participant a, for some a. Assume that none of the first three cases applies. Let the two conversations of a be $\{a, b\} \prec \{a, c\}$. Let $\{b, d\}$ denote the b-successor of $\{a, b\}$, and let $\{c, e\}$ denote the c-predecessor of $\{a, c\}$. As $\{a, b\}$ is also b's first conversation, and $\{a, c\}$ is c's final conversation, the secret of a can only reach $\{1, \ldots, n\} \setminus \{a, b, c\}$ via $\{b, d\}$, which takes at least $n - 3$ conversations. Similarly, we need at least $n - 3$ conversations to collect the secrets of $\{1, \ldots, n\} \setminus \{a, b, c\}$ in conversation $\{c, e\}$. If these sets of conversations are disjoint we have at least $2 + 2(n - 3) = 2n - 4$ conversations and we are done. If they are not disjoint, then $\{b, d\} \rightsquigarrow \{c, e\}$, in other words, the secrets of a and b reach c via conversation $\{c, e\}$. But then case 3 applies.

Case 5. Each participant is involved in at least four conversations. Then obviously, the number of conversations is half of the number of conversation-participant combinations, which is at least half of $4n$. Hence, in this case $|G_n| \geq 2n \geq 2n - 4$.

Case 6. Each participant is involved in at least three conversations, participant a is involved in exactly three conversations. If the first five cases do not apply, this last one must apply, for some a.
Let a participate in $\{a, b\} \prec \{a, c\} \prec \{a, d\}$. Let $\{b, b'\}$ directly succeed $\{a, b\}$; let $\{c, c'\}$ directly precede $\{a, c\}$, and $\{c, c''\}$ directly succeed $\{a, c\}$; let $\{d, d'\}$ directly precede $\{a, d\}$. See figure 1, where B and C'' denote the sets of conversations reached from $\{a, b\}$, $\{a, c\}$, and C' and D the sets of conversations leading to $\{a, c\}$ and $\{a, d\}$, respectively, via the partners of a.

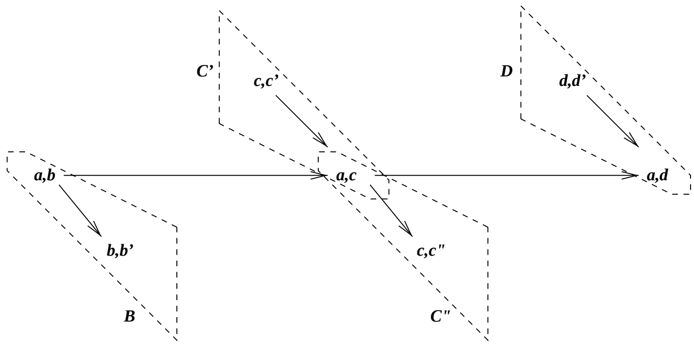

Figure 1 Conversations reached by or reaching a

We first argue that these sets are disjoint (except for $C' \cap C'' = \{a, c\}$). For suppose they are not disjoint. If $\{b, b'\} \rightsquigarrow \{c, c'\}$, delete conversations $\{a, b\}$ and $\{a, d\}$, and change $\{a, c\}$ to $\{c, d\}$. We then obtain a sequencing of conversations in which $n - 1$ secrets are exchanged, and conclude that $|G_n| \geq 2 + \phi_{n-1} \geq 2n - 4$. If $\{c, c''\} \rightsquigarrow \{d, d'\}$, then the previous argument can be repeated, by considering the reversed sequence $\overleftarrow{G}_n$.

If $\{b, b'\} \rightsquigarrow \{d, d'\}$ and $\{b, b'\} \not\rightsquigarrow \{c, c'\}$, then we may assume without loss of generality, that $\{a, c\}$ is timed before $\{b, b'\}$. Delete $\{a, b\}$ and $\{a, d\}$, and replace $\{a, c\}$ by $\{b, c\}$. Again, we obtain a sequence of conversations in which $n-1$ secrets are exchanged, and conclude that $|G_n| \geq 2 + \phi_{n-1} \geq 2n-4$.

From now on we may assume that the sets are disjoint. Observe that $|B \cup C''| \geq n-2$, since it takes at least $n-1$ conversations to spread the secret of a, and only conversation $\{a, d\}$ is not contained in $B \cup C''$.
By considering the reversed sequence $\overleftarrow{G}_n$, a similar argument shows that $|D \cup C'| \geq n-2$. So we are almost done, since we found by now, that $|B \cup C' \cup C'' \cup D| = |B \cup C''| + |C' \cup D| - 1 \geq 2n-5$.
We finally claim that $|B \cup C''| \geq n-1$ or $|D \cup C'| \geq n$. It follows from the proof above, that $|B \cup C''| = n-2$ **only** in the case that in each conversation one person learns the secret of a. This happens in particular in the final conversations in $B \cup C''$. There are $(n-2)/2$ of these, since $\{a, d\}$ is the only final conversation not contained in $B \cup C''$.
Consider a final conversation $\{q, r\} \in B \cup C''$, and let q be the participant that learns secret a. Then the q-predecessor $\{p, q\}$ of $\{q, r\}$ must belong to $D \cup C'$. Since q makes at least three conversations, $\{p, q\}$ cannot be a first conversation. Note that the p-successor of $\{p, q\}$ must belong to $C' \cup D$.
By the reasoning above we find for each final conversation in $B \cup C''$, a distinct non-first conversation in $D \cup C'$. As a consequence $D \cup C'$ contains $(n-2)/2$ first conversations and at least $(n-2)/2$ non-first conversations plus two conversations with participant a, hence in total $|D \cup C'| \geq n$.

We now finally see that $|G_n| = |B \cup C''| + |D \cup C'| - 1 \geq 2n-2-1 = 2n-3$. This last case settles the proof of our claim. ⇠

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