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How to recognize functions in $L_p(\mathbf{R}) + L_q(\mathbf{R})$

Consider two function spaces $L_p = L_p(\mathbf{R})$ and $L_q = L_q(\mathbf{R})$ $(0 < p \le q \le \infty)$. The interest in the space $L_p + L_q = \{f = f_p + f_q : f_p \in L_p, f_q \in L_q\}$ originates in Fourier analysis [2 (p.18)]: for any function $f = f_1 + f_2 \in L_1 + L_2$ it is possible to define a Fourier transform $\hat{f} = \hat{f}_1 + \hat{f}_2 \in C_0 + L_2$ and $\hat{f}$ is well-defined even if the representation of $f = f_1 + f_2$ is not unique.

This definition extends the Fourier transform to all functions $f \in L_s$ $(1 \le s \le 2)$, since it is easy to see that $L_s \subset L_p + L_q$ for $p < s < q$ [**1** (13.19)]; for any $f \in L_s$ we have

$$\int_{\{|f|>1\}} |f|^p dx \le \int_{\{|f|>1\}} |f|^s dx < \infty,$$

$$\int_{\{|f|\le 1\}} |f|^q dx \le \int_{\{|f|\le 1\}} |f|^s dx < \infty \quad \text{if } q < \infty,$$

$$f = f1_{\{|f|>1\}} + f1_{\{|f|\le 1\}} \in L_p + L_q. \tag{1}$$

Not every function in $L_p + L_q$, however, needs to belong to some space L_s, as demonstrated by the functions $f \in L_1 + L_2$ defined by $f(x) = x^\alpha$ for $\alpha \in]-1, -\frac{1}{2}[$.

If one wants to apply a Fourier transformation $\hat{f} = \hat{f}_1 + \hat{f}_2$ to a given function f on $\mathbf{R}$, one has to make sure that f belongs to $L_1 + L_2$ and one has to exhibit the components f_1 and f_2 of some representation of f as in (1). Since in general $|f_p + f_q|$ may be small if $|f_p|$ and $|f_q|$ are both large and either of these may be small if $|f_p + f_q|$ is large it is not obvious that in general the functions

$$f_> = f1_{\{|f|>1\}} \quad \text{and} \quad f_< = f1_{\{|f|\le 1\}}$$

serve to determine indices p and q and furnish a decomposition as in (1). Concerning the latter remark we have the following theorem.

Theorem. *A complex-valued function f belongs to $L_p + L_q (0 < p \le q \le \infty)$ if and only if $f_> \in L_p$ and $f_< \in L_q$.*

Proof. Since $f = f1_{\{|f|>1\}} + f1_{\{|f|\le 1\}}$ the 'if'-part is clear.

Conversely, if $f = f_p + f_q$ ($f_p \in L_p$, $f_q \in L_q$ without loss of generality we assume $0 < p \le q \le \infty$), then we have

$$\begin{aligned}|f|1_{\{|f|>1\}} &\le |f|1_{\{|f_p|>\frac{1}{2}\}} + |f|1_{\{|f_q|>\frac{1}{2}\}} \\ &\le |f_p|1_{\{|f_p|>\frac{1}{2}\}} + |f_q|1_{\{|f_p|>\frac{1}{2}\}} + |f_p|1_{\{|f_q|>\frac{1}{2}\}} + |f_q|1_{\{|f_q|>\frac{1}{2}\}}.\end{aligned} \tag{2}$$

Since the sets $\{|f_p| > \frac{1}{2}\}$ and $\{|f_q| > \frac{1}{2}\}$ have finite measure, all four functions on the right side of (2) and therefore also $f1_{\{|f|>1\}}$ belong to L_p. Furthermore,

$$\begin{aligned}|f|1_{\{|f|\le 1\}} &\le (|f_p| + |f_q|)1_{\{|f_p|\le 1, |f_q|\le 1\}} + 1_{\{|f_p|>1\}} + 1_{\{|f_q|>1\}} \\ &\le |f_p|1_{\{|f_p|\le 1\}} + |f_q|1_{\{|f_q|\le 1\}} + 1_{\{|f_p|>1\}} + 1_{\{|f_q|>1\}}.\end{aligned} \tag{3}$$

Again all four functions on the right side of (3) belong to L_q, therefore also $f1_{\{|f|\le 1\}}$. □

Since for a given function f on $\mathbf{R}$ the integrals $\int_{\{|f|>1\}} |f|^s dx$ and $\int_{\{|f|\le 1\}} |f|^s dx$ are monotone increasing respectively decreasing functions of s we obtain $L_p + L_q \subset L_{p'} + L_{q'}$ for $0 < p' \le p \le q \le q' \le \infty$. For a given function f on $\mathbf{R}$ having the property that $f_> \in L_p$ and $f_< \in L_q$ $(0 < p \le q \le \infty)$ define

$$\overline{p} = \sup\{p : f_> \in L_p\}, \quad \overline{q} = \inf\{q > 0 : f_< \in L_q\}.$$

Then, for finite $\overline{p}$ respectively $\overline{q}$, the integrals $\int_{\{|f|>1\}} |f|^{\overline{p}} dx$ and $\int_{\{|f|\le 1\}} |f|^{\overline{q}} dx$ may be finite or not [1 (13.28)].

As a consequence of the theorem we obtain the following corollary:

Corollary. *If $\overline{p} > \overline{q}$ then $f \in L_s$ for all $s \in]\overline{q}, \overline{p}[$. If $\overline{p} \le \overline{q}$ then $f \in L_p + L_q$ for all $p < \overline{p}$ and all $q > \overline{q}$. If $\overline{p} < \overline{q}$ then $f \notin L_s$ for all $s > 0$.*

The mentioned statements can be carried over to functions on a σ-finite, infinite non-atomic measure space.

References

1 Hewitt, Edwin/Stromberg, Karl: *Real and Abstract Analysis*. Springer Verlag Berlin Heidelberg New York, 1965.

2 Stein, Elias M./Weiss, Guido: *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, 1971.