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The Smarandache harmonic series

For every positive integer n let $S(n)$ be the minimal positive integer m such that $n \mid m!$ This function is known as the Smarandache function. We begin with a couple of considerations concerning the function S. First of all, let us notice that if n is a squarefree number, say $n = q_1 q_2 ... q_t$, where $2 \leq q_1 < ... < q_t$ are prime numbers, then $S(n) = q_t$. Secondly, let us notice that $\lim_{n \to \infty} S(n) = \infty$. Indeed, this equality follows right-away by noticing that if k is a positive integer and n is a positive integer such that $S(n) \leq k$, then $n \leq k!$

In this note, we analyze convergence questions for some series of the form

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}, \tag{1}$$

or close variations of it.

Divergent Series

In this section, we point out that

Theorem 1. *For any $\delta < 1$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{(\log n)^{\delta}}} \tag{2}$$

diverges.

Theorem 1 has the obvious

Corollary 1. *Let $\delta > 0$. Then, series (1) diverges. Moreover, the series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{(\log \log n)^{\delta}}} \tag{3}$$

diverges as well.

Proof of theorem 1. For any $t \geq 1$ let $p_1 < p_2 < ... < p_t$ be the first t primes. By the remarks made in the Introduction, we know that any number of the form $n = p_t m$ where m is squarefree whose prime factors are among the numbers $p_1, ..., p_{t-1}$ will obviously satisfy $S(n) = p_t$. Since there are exactly 2^{t-1} such numbers (that is, the powerset of $\{p_1, p_2, ..., p_{t-1}\}$) and since each one of them is smaller than p_t^t, it follows that series (2) is bounded from below by the subseries

$$\sum_{t \geq 1} \frac{2^{t-1}}{p_t^{(t \log(p_t))^{\delta}}} = \sum_{t \geq 1} 2^{t-1-(t \log(p_t))^{\delta} \log_2 p_t}. \tag{4}$$

Since by the prime number theorem

$$\lim_{t \to \infty} \frac{p_t}{t \log t} = 1, \tag{5}$$

and $\delta < 1$, it follows immediately that

$$t - 1 - (t \log(p_t))^{\delta} \log_2 p_t > 0$$

for t large enough. In particular, the general term of (4) is unbounded, which certainly implies that (4) is divergent. □

One can use Dirichlet's theorem on the of size of the t-th prime in an arithmetical progression (see page 247 in [1]) to show that the series (1)–(3) remain divergent if instead of summing over all the positive integers one sums only over all the terms of a fixed arithmetical progression $(ak + b)_{k \geq 1}$.

Convergent Series

In this section, we mention some convergent series involving the function S.

Theorem 2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{S(n)^{\delta}}} \tag{6}$$

converges for all $\delta \geq 1$ and diverges for all $\delta < 1$.

Theorem 3. *For any $\epsilon > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon \log n}}$$

converges.

It is unclear to us how Theorem 2 relates to Theorem 3.

Proof of theorem 2. We treat the case $\delta < 1$ first. Here, the arguments employed in the proof of Theorem 1 show that series (6) is bounded below by

$$\sum_{t\geq 1} \frac{2^{t-1}}{p_t^{p_t^{\delta}}} = \sum_{t\geq 1} 2^{t-1-p_t^{\delta}\log_2 p_t}. \tag{7}$$

Since $\delta < 1$ it follows, by the limit (5), that $t-1-p_t^{\delta}\log_2 p_t > 0$ for t large enough, which rules out the convergence of (7).
We now assume that $\delta = 1$. We show something stronger, namely that

$$\sum_{n\geq 1} \frac{1}{S(n)^{\epsilon S(n)}} \tag{8}$$

converges for all $\epsilon > 0$. It certainly suffices to assume that $\epsilon \leq 1$. Series (8) can be rewritten as

$$\sum_{k\geq 1} \frac{u(k)}{k^{\epsilon k}}$$

where $u(k) = \#\{n \mid S(n) = k\}$. Since every n such that $S(n) = k$ is a divisor of $k!$, it follows that

$$u(k) \leq d(k!).$$

By formula (1) on page 111 of [1], we know that $d(l) < Cl^{\epsilon}$ for any positive integer l, where C is some constant (depending on ϵ). Hence,

$$u(k) \leq d(k!) < C(k!)^{\epsilon} < C_1(k/2)^{\epsilon k} \tag{9}$$

for some constant C_1 (the last inequality in (9) follows from Stirling's formula). From (9), it follows that series (8) is bounded above by

$$C_1 \sum_{k\geq 1} \frac{1}{k^{\epsilon k}} \cdot \left(\frac{k}{2}\right)^{\epsilon k} = C_1 \sum_{k\geq 1} \frac{1}{2^{\epsilon k}} = \frac{C_1}{2^{\epsilon}-1}.$$

□

Proof of theorem 3. We make the argument first in the case $\epsilon = 1$ and then we explain how the argument can be adapted to the general case.
We begin by excluding the even numbers. Every even number is either a power of 2, or it is divisible by an odd number > 1. Let us first account for the contributions of the powers of 2. When $n = 2^{\beta}$, it follows easily that $S(n) \geq \beta$. Hence, these contributions are bounded above by

$$\sum_{\beta\geq 1} \frac{1}{\beta^{\beta \log 2}}$$

which is obviously convergent. Assume now that $n = 2^{\beta}m$ for some $m > 1$. Since $S(n) \geq S(m)$, it follows that the contributions of all the numbers of the form $2^{\beta}m$ for some $\beta \geq 1$ are bounded above by

$$\sum_{\beta\geq 1} \frac{1}{S(m)^{(\log m + \beta \log 2)}} = \frac{1}{S(m)^{\log m}} \sum_{\beta\geq 1} \frac{1}{S(m)^{\beta \log 2}}$$
$$= \frac{1}{S(m)^{\log m}} \frac{1}{S(m)^{\log 2}-1} \leq \frac{C}{S(m)^{\log m}},$$

where $C = \frac{1}{3^{\log 2}-1}$. Hence, it suffices to look at the series

$$\sum_{m \text{ odd}} \frac{1}{S(m)^{\log m}}. \tag{10}$$

It is clear that for any integer m, $S(m)$ is divisible with at least one of the primes p dividing m. Fix such a prime p and look at all the possible integers m whose S is a multiple of p. Clearly, $S(m) \geq p$ and $m = pu$ for some integer u. Let us count the u's now. For every $s \geq 0$, there are at most $e^{s+1} - e^s + 1$ integers u in the interval $[e^s, e^{s+1})$ and each one of them will satisfy $\log u \geq s$. Hence, for p fixed, the contributions of all those m's is at most

$$\sum_{s\geq 0} \frac{e^{s+1}-e^s+1}{p^{\log p + s}} < \frac{1}{p^{\log p}} \sum_{s\geq 0} \frac{e^{s+1}}{p^s} = \frac{e}{p^{\log p}} \cdot \frac{1}{1-e/p} < \frac{C}{p^{\log p}}, \tag{11}$$

where $C = \frac{e}{1-e/3}$. Hence, series (10) is bounded above by

$$C \sum_{p \text{ prime}} \frac{1}{p^{\log p}}$$

which is obviously convergent.

Suppose now that $\epsilon < 1$ is arbitrary. Then one applies the procedure outlined at the beginning of the argument and eliminates, one by one, all primes p such that $p^{\epsilon} < e$. Once this has been achieved, then one can apply the argument explained above in the case m odd. Indeed, the reason why this argument worked is because series (11) is geometric with ratio e/p smaller than 1 (notice that (11) wouldn't have worked out for $p = 2$ because $2 < e$). At the end, one obtains just the series

$$\sum_{p \text{ prime}} \frac{1}{p^{\epsilon \log p}}$$

which is obviously convergent. □

References

1 L.K. Hua, *Introduction to Number Theory*, Springer-Verlag, 1982.