B. Mond
Department of Mathematics, La Trobe University, Bundoora, Vic. 3083, Australia
B.Mond@latrobe.edu.au

J. Pečarić
Faculty of Textile Technology, University of Zagreb, Zagreb, Croatia
Applied Mathematics Department, University of Adelaide, Adelaide, South Australia, 5005, Australia
jpecaric@maths.adelaide.edu.au

Inequalities for exponential functions and means, II

An inequality involving exponential functions was used to establish corresponding inequalities for means for both real numbers and matrices. Here we give an extension and simpler proof of this inequality.

The following bounds for the difference between the weighted arithmetic and geometric means was given by Alzer [1] (see also [2, p.38]):

$$0 \le Ae^{-G/A} - Ge^{-A/G} \le \frac{3}{e}(A-G). \tag{1}$$

In [3], it was noted that the improved result

$$0 \le C - D \le Ce^{-D/C} - De^{-C/D} \le \frac{3}{e}(C-D) \tag{2}$$

holds for arbitrary positive real numbers C and D such that $C \le D$ (not necessarily the weighted arithmetic and geometric means). Thus, (1) and (2) are essentially exponential results rather than results for means, which are special cases. In particular, instead of the arithmetic and geometric means, we can use arbitrary comparable means [4].

In [3], means of matrices were also considered, i.e., it was noted that (2) holds for Hermitian positive definite matrices C and D such that $C \ge D$ where $C \ge D$ means that $C - D$ is positive semidefinite.

In the proof of (2) (as well as in the proof of the matrix case) the main result used was a corresponding inequality for real numbers, that is, if $x \in [0,1]$, then

$$1 - x \le e^{-x} - xe^{-1/x} \le \frac{3}{e}(1-x). \tag{3}$$

The values $3/e$ on the right and 1 on the left cannot be replaced by a smaller number on the right or a larger number on the left. Equalities hold in (3) if and only if $x = 1$.

The following extension of (3) holds:

Theorem. *If $x, y \in (0,1)$ with $x < y$, then*

$$1 < \frac{e^{-x} - xe^{-1/x}}{1-x} < \frac{e^{-y} - ye^{-1/y}}{1-y} < \frac{3}{e}. \tag{4}$$

Proof. The second inequality shows us that the function

$$f(x) = \frac{e^{-x} - xe^{-1/x}}{1-x}, \ x \in (0,1)$$

is an increasing function. Indeed, we have

$$f'(x) = \frac{g(x)}{x(1-x)^2}, \tag{5}$$

where
$$g(x) = x^2 e^{-x} - e^{-1/x}.$$

Let us note that the function
$$h(x) = 2\ln x - x + 1/x$$

has the same sign as $g(x)$. Further, we have $h(1) = 0$ and
$$h'(x) = -\frac{(x-1)^2}{x^2}.$$

Thus, it is obvious that $h'(x) < 0$ and so h is a decreasing function on $(0,1)$. Hence h is positive on the same interval and therefore g is positive. Now from (5), we have that $f(x)$ is an increasing function.

The first and third inequalities in (4) now follow from the following result given in [3]:
$$\lim_{x \to 1} \frac{e^{-x} - xe^{-1/x}}{1-x} = \frac{3}{e} \quad \text{and} \quad \lim_{x \to 0+} \frac{e^{-x} - xe^{-1/x}}{1-x} = 1.$$

□

Remark. It is clear that (3) is a special case of (4). Moreover, note that the above proof is simpler than that given in [2].

Corollary 1. *Let A and G be weighted arithmetic and geometric means of real numbers $x_i (i = 1, \ldots, n)$ such that $x_i \in (0,1)$, $i = 1, \ldots, n$. Then*
$$1 < \frac{e^{-G} - Ge^{-1/G}}{1-G} \le \frac{e^{-A} - Ae^{-1/A}}{1-A} < \frac{3}{e}.$$

Proof. It is clear that $0 < G \le A < 1$ so that the Theorem is applicable. □

Corollary 2. *If A, B, C, D are positive numbers such that $A/B < C/D < 1$, then*
$$1 < \frac{Be^{-A/B} - Ae^{-B/A}}{B-A} < \frac{De^{-C/D} - Ce^{-D/C}}{D-C} < \frac{3}{e}. \tag{6}$$

Proof. The result is a simple consequence of the Theorem for $x = A/B$, $y = C/D$. □

Corollary 3. *If $0 < x_i \le \frac{1}{2}$ for $i = 1, 2, \ldots, n$, not all x_i equal, then (6) still holds with either*

(i)
$$A = \prod_{i=1}^n x_i,\ B = \prod_{i=1}^n (1-x_i),$$
$$C = \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^n,\ D = \left[\frac{1}{n}\sum_{i=1}^n (1-x_i)\right]^n,$$

or

(ii)
$$A = \prod_{i=1}^n x_i,\ B = \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^n,$$
$$C = \prod_{i=1}^n (1-x_i),\ D = \left[\frac{1}{n}\sum_{i=1}^n (1-x_i)\right]^n.$$

Proof. Both results are simple consequences of Corollary 2, the arithmetic-geometric mean inequality, and the well-known Ky Fan inequality ([5; p.5]):
$$\frac{\prod_{i=1}^n x_i}{\left(\sum_{i=1}^n x_i\right)^n} \le \frac{\prod_{i=1}^n (1-x_i)}{\left[\sum_{i=1}^n (1-x_i)\right]^n} \tag{7}$$

with equality in (7) holding only if all x_i are equal. □

Remark. Of course, if we write (7) in means form
$$\frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{\frac{1}{n}\sum_{i=1}^n x_i} \le \frac{\left(\prod_{i=1}^n (1-x_i)\right)^{1/n}}{\frac{1}{n}\sum_{i=1}^n (1-x_i)}, \tag{7'}$$

we can give some further related results as applications of (6) and (7′).

References

1 H. Alzer, *A lower bound for the difference between the arithmetic and geometric means*, Nieuw Archief voor Wiskunde, **8** (1990), 195–197.

2 D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Norwell, MA, 1993.

3 M. Alić, B. Mond, J. Pečarić and V. Volenec, *Bounds for the differences of matrix means*, SIAM J. Matrix Anal. Appl., **18** (1997), 119–123.

4 M. Alić, B. Mond, J. Pečarić and V. Volenec, *Inequalities for exponential functions and means*, Nieuw Archief voor Wiskunde, **14** (1996), 343–348.

5 E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, New York, 1965.