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Johann Bernoulli lecture, Groningen, 20 April 1999

Mathematical platonism reconsidered

This paper discusses current challenges to the standard Platonist philosophy of mathematics. Evidence from neurophysiology and comparison with computers reveal that the human way of conceiving mathematics bears strong marks of the specific structure of our brain, and of its shortcomings. This does not fit well with a universalist view of the conceptual nature of mathematics. Another worry comes from discussing the physical status of information in the presence of quantum mechanics and general relativity. Again, this may not fit well with the very classical Platonist Weltanschauung.

Plato's dialogues present us with a world of pure ideas, to which the philosophers have access, and from which they can come back to teach the rest of mankind who remain locked up in a dark cavern. Nothing fits this description better than the world of mathematics, with its unchangeable truths, and its servants, the philosopher-mathematicians. Admittedly, the mathematician's ideas reside in a modest amount of jelly-like substance which constitutes the mathematician's brain, but this is irrelevant to the eternal truth that 7 is a prime number, or that there are infinitely many primes.

The Platonist view of mathematics is challenged now and then from different sides. There has been the crisis of the foundations at the beginning of this century, and Gödel's theorem. Many mathematicians think that these problems are well clarified, and need not bother us too much. There is however also the problem of the physical basis for mathematical thinking and computing, whether it is the mathematician's brain or a computing machine. This problem recurs in various forms and is not likely to disappear readily. It was easy for the Greeks to imagine a world of pure ideas, which the philosophers shared with the Gods. The Gods have gone away, and we now converse with electronic computers which are more real and effective, but perhaps not as good company. What kind of company they are, and what they teach us about ourselves, is one of the things I would like to discuss. I shall adopt the point of view of a modern counterpart of the Greek philosopher, namely today's working mathematician, or more generally scientist. And I shall start with a very concrete example.

Computer-aided proofs

My colleague Oscar Lanford has spent time, among other things, on computer-aided proofs. What are these? Well, for some (interesting) mathematical theorems there is a relatively straightforward and natural way of obtaining a proof, which consists in making precise numerical checks. The calculations needed for the numerical check may be so extremely long that one cannot realistically perform them by hand. One then uses a computer, and obtains thus a computer-aided proof. I shall now briefly sketch the sort of problem analyzed by Lanford [8], to convince working mathematicians that his approach is indeed straightforward and natural. Non-mathematicians can skip over the few technical details that follow.

Lanford is currently working on an analogue of the Feigenbaum fixed point problem. Specifically, he wants to prove that a certain map F in some function space E has a fixed point x, *i.e.*, $Fx = x$. There is convincing numerical evidence that the fixed point x does indeed exist and that F is hyperbolic at x. Hyperbolicity, roughly speaking, means that near the fixed point x, F expands in a certain direction and contracts in a complementary direction. If x_0 is an approximate fixed point of F, *i.e.*, Fx_0 is suitably close to x_0, and if F is hyperbolic near x_0, a general theorem asserts that F has a unique exact fixed point close to x_0. But it is known that the hyperbolicity of F can be checked by calculations of finite precision. In brief, the theorem one wants to prove, namely the existence of a fixed point, can be obtained by calculations of finite precision. This does not mean that things are easy: one has to find a basis for the infinite dimensional space E, calculate with vectors which have a finite (but possibly large) number of components, and keep a strict record of the errors committed. The bookkeeping of errors is helped by so-called interval arithmetics, where each real number is replaced by an interval that contains it, and intervals are combined in the various operations on reals so that for each quantity one knows an explicit interval which certainly contains this quantity. Computers can do interval arithmetics, and I shall not discuss here the problem of making sure that the machine really does what it claims to do.

Having described the mathematical principle of Lanford's computer-aided proof, let me present some practical details. The computer program which will prove the theorem (when fed into the computer) is about 150 pages long, and written in a variant of the C language which allows a step by step explanation of what the program is doing. (Long programs tend to be quite impenetrable, and this is after all supposed to be a *proof*, which a human mathematician can check, even if the aid of a computer is also needed). You now load your program in the computer, push ENTER, and there comes a message on the screen saying "YEP! your theorem is proved" (or "NOPE! try again"). I think I have shown to you that use of a computer is a natural and reasonable way of proving certain types of theorems. It takes a couple of years of work, and that seems to be the end of the story, but it ain't...

At the end of the seminar where he presented his work, Lanford made some comments which I shall now relate, and which you may find deeply disturbing. "I am absolutely certain" says Lanford "that there are mistakes in my 150 page program." Now, isn't this deeply disturbing? You thought your theorem was proved, but reading the program once more you find a mistake, fix it, run the program again, and the answer now is "NOPE! try again". It would seem that this discredits the whole approach. But I suggest that you suspend your judgment until you hear the further comments of Lanford who is, by the way, an excellent mathematician and also, by all standards, an extremely cautious one. "I am absolutely certain" says Lanford "that the theorem is true." This is because of the extensive numerical checks that he made before embarking in his computer-aided proof. "And I am quite confident that any error that is found in my program can be fixed so that the proof works." This is all well and fine, but wouldn't you rather have an old style computer-less proof that you can really trust? This is debatable because, although our mathematical proofs should in principle be formalizable, they are in fact not formalized. Formalization would be a formidable task. It would entail a lot of explaining "abuses of language", filling gaping holes in the logics, translating into incomprehensible formal language, and debugging: as of today this is simply not done.

As Oscar Lanford points out[1], there are mechanical ways to flush out errors in computer programs, and these are lacking for traditional mathematics. On the other hand "mathematical proofs – precisely because they are intended to be understood by our organic, evolution-conditioned brains – have a coherence and capacity for error-correction that programs lack". In conclusion, Lanford wants to avoid comparing the frequency of errors in programs and in standard mathematics, a question about which he says he has not made up his mind.

Idiosyncrasies of human mathematics

If human mathematics isn't really formal mathematics, what is it? I have been daydreaming about meeting a mathematician from outer space and comparing notes with him, or her, or it (see [10], [11]). I have eventually come to the conclusion that the mathematician from outer space had the form of a pleasant young woman. (If you prefer a Greek God, we can compromise on a Greek Goddess). Her name is Pallas, and she is doing research on human mathematics. Her theory is that human mathematics is rather peculiar as compared with the mathematics of other mathematically competent species of the Galaxy, and that our peculiarities are due to the idiosyncratic shortcomings of the human brain (see [11]).

Of course you may exclaim that Pallas is just a figment of my imagination, that we know no other mathematically competent species of the Galaxy, and that speculations on their mathematics is therefore unfounded and worthless. I agree with that only in part. Our electronic computers are not good mathematicians, but they have some mathematical competence, and there are certain mathematically useful things that they do much better than humans. (Remember that Gauss and Riemann did extensive numerical computations by hand, and that today's mathematicians often do the same using their computers). The conclusion is unescapable that there are some useful talents that we might possess but don't, and that the lack of these talents may have some deep effect on human mathematical achievements.

It happens to be feasible and fruitful to evaluate some of the abilities and shortcomings of the human mathematical brain, using neurophysiology and comparison with computers. In what follows we list and analyze our findings. The first discussion of this type can be found in J. von Neumann's book *The computer and the brain* [9]. There is by the way no reason to be outraged by the comparison between computer and brain: the two turn out to be rather different in their details, but they are both information processing machines, and have therefore things in common, like the need for memory. The comparison is thus enlightening, here it goes.

The architecture of the brain is highly parallel.

The number of neurons in the brain is huge ($> 10^{10}$), and their organization is highly parallel. Large number of components and parallelism may also be present in computers, but not to the same amount.

The brain is slow.

Because of the relative slowness of the nervous influx, characteristic times in the nervous systems are much larger than in computers. In agreement with this, computers typically perform repetitive calculations where each loop provides an updated input for the next loop. The brain by contrast often uses its high parallelism to treat information (for instance visual information) in a direct way, without using loops. When however we reflect upon some question, say mathematical, we probably use the same brain circuits repeatedly, as the word 'reflect'

1 I am here quoting from an e-mail message sent by Lanford.

suggests, and it takes a lot of time.

Our memory is poor.

In particular our short term memory is typically limited to about seven items, which is catastrophic. This can be remedied to some extent by "memorizing", *i.e.*, putting in long term memory. (There is much more room there, but memorizing takes time and effort).

Our mathematical thinking uses various systems of the brain: vision, language, etc.

For many people, language seems to be the very essence of thinking, but Einstein notes that in his case the main elements of thinking are "of visual and some muscular type". Using the visual system is important for most mathematicians, and "geometrization" of a theory is considered a great achievement. Obviously we use, for doing mathematics, parts of the brain that were developed by evolution for other purposes. The result is not bad, but strongly bears the marks of its origins: it is expressed in an informal "natural" language, and makes strong appeal to the visual system.

We can focus our attention on a task, but in a limited way.

We like to point out that we differ from computers in having consciousness. Therefore, it is a bit disturbing that some mathematical work seems to be done unconsciously, as noted by Poincaré. In fact we are unable to define consciousness, but it is related to attention, which is a human ability to concentrate intellectual resources on some item at a certain time. (Because of the high parallelism of the brain, there are always many other things "running in the background"). Our attention span is limited, as is our short term memory, and this has consequences that we discuss as our next two items.

We like short formulations.

What we call mathematical elegance is the simplicity that underlies some very complicated problems. The simplicity may be in the formulation of a result (Fermat's last theorem), or in the ideas behind a theory. One can understand the appeal of such simplicity to a human mind so limited in its attention span and its memory.

We are not good at formal logical manipulations.

While mathematical proofs are in principle formalizable, we stay away from formal mathematical texts, even though they could be checked mechanically to be correct. We are not good at such checks, because of our poor memory and attention span.

But we are quite good at finding regularities, or 'meaning'.

After a long list of features for which the human brain is inferior to a stupid PC, here is finally something that we seem to do better. The human brain is untiring at "interpreting" the data it receives. This can border on the ridiculous when we look for arithmetic regularities in the digits of a telephone number that we remember with difficulty due to our poor memory. But our unflagging "search for meaning" is probably what underlies the mathematical ability of the human mind. Note that finding meaning and regularities has obvious survival value, and has been encouraged by natural selection, while remembering telephone numbers has not been so favored. Note also that discovering regularities is not a very clearly defined task. If it were, we could program it on a computer, and the computer might get better at doing mathematics than we are.

How do we do mathematics?

We do not clearly understand how we function as mathematicians, otherwise we could write a computer program to do the same job. (I think that such a program will some day be written, leaving us out of business, and I am not looking forward to it). We can however give an impressionistic view of the working mathematical brain, compatible with the features we have discussed above.

For most of us scientists, the mother tongue is different from English, which is our professional language. Registering the activity of the mathematician's brain would thus give phrases or bits of sentences in two languages with somewhat different roles (perhaps remarks and expletives in the mother tongue, and technical stuff in English). But the verbal output would be interspersed with nonverbal elements, visual for most of us (a glimpse of a triangular matrix, or of Poincaré's bearded face). Because of poor memory and limited attention span, we use abbreviated verbal or nonverbal symbols and try to combine them into something useful. The creative work is thus of combinatorial nature, using analogy for guidance. To relieve our poor memory we often take a piece of paper and draw diagrams or scribble formulas: the piece of paper plays the role of an external memory and makes good use of our visual competence. Historically, the first high-level mathematical theory was Greek geometry where figures are fundamental, using visual intuition and serving as external memory in an incredibly effective manner. The next great intellectual explosion in mathematics was based on the manipulation of formulas, using again an external memory of visual nature, but with wider ranging applicability than the drawings of Greek geometry. Symbolic manipulation of formulas goes to the heart of what we consider mathematics, but is getting far from what our brain has been prepared for by natural evolution. We idealize mathematics as consisting of texts in formalized language, but we do not write such texts, and if they were written we would be unable to read them. Let us leave idealized formal texts and come back to human mathematics. After some time, the problem which we investigate has become familiar, *i.e.*, things have been put in long term memory, and we can do significant work without our piece of paper. In fact creative work can be done unconsciously, as noted by Poincaré. And then, more or less suddenly we are convinced that we have a good idea. But because of the way we function the idea is in terms of abbreviated symbols. Something like this: "*it works* because *there is* a fixed point and *if you look at it the right way* it is more or less clear that it is unique". (A sentence of this sort is common towards the end of a seminar lecture and, depending on the case, makes everything clear or leaves you in the deepest fog). We have now to unpack the abbreviated symbols, hope that there won't be a really bad surprise and write things out in sufficient details that colleagues will be convinced.

I hope that the above description of mathematical work has sounded sufficiently familiar and convincing (even though I have left out much, like the initial planning of a strategy). Does this all fit a Platonist view of mathematics? In a sense, yes, because the mathematician's world is a world of ideas as envisioned by Plato. But what comes out of our discussion is that these ideas are very specifically human, depending on the very special organization of our brain, and in particular on its shortcomings. The truth value of specific statements is something that we share with the Gods, but what we see as the deep underlying ideas "which make things work" may just be the invention of the mortals that we are. In other words, the high price that we put on *conceptual aspects* of mathematical theories may just reflect our specifically human limitations with regard to memory and attention span.

How is mathematical truth anchored in physical reality?

Doing mathematics consists in manipulating information. Specifically, formalized mathematics operates on strings of symbols. How seriously should one take the fact that information and its manipulation take place in the physical world? The Platonist view would be that this is not important. Plato might in fact have found nothing wrong with the Turing machine, and its infinite ribbon of paper. One may worry that, if our physical universe is finite, it cannot contain an infinite ribbon of paper. More generally, the proofs of certain theorems may be so cumbersome that they do not fit in our physical universe. The truth of such theorems would thus be inaccessible to us, which is annoying, but again does not threaten a Platonist view of mathematics: at least the Gods would know.

If one considers seriously the physical task of implementing algorithms, writing proofs or making calculations in the physical world of the 20-th century, one meets a number of problems. Here are a few that have been recognized.

1 Information has a physical basis, and the energy costs of making calculations needs to be discussed (see Landauer [7]).
2 Since information has a physical basis, one can ask how much can be packed in a given ball. The answer is apparently not proportional to the volume of the ball, but to its surface (see Beckenstein [2], [3]). This is in agreement with the Beckenstein-Hawking formula that the entropy of a black hole is proportional to its surface (general relativity also indicates that if you put too much in a small region, you create a black hole).
3 Our world obeys the laws of quantum mechanics, and it makes sense to try to construct computers using quantum interference effects. While this is difficult to do in practice, it is known in principle that quantum computers could solve certain problems much more efficiently than classical computers. (For reviews see for instance Aharonov [1], Kitaev [6]). It appears however that our brain does not use quantum computing (see Hepp [4]).

The fundamental limitations put by physical law on computing, or doing mathematics, do not appear to be very well understood at this time. It is thus reasonable for me to stop here my discussion. It seems possible, however, that another crisis of foundations of mathematics may be awaiting us, and that collision with physical law could cause further damage to our Platonist conception of mathematics.

Acknowledgments

I am very thankful to Karine Chemla and Oscar Lanford for critical discussions of the manuscript, and to Thibault Damour and Klaus Hepp for useful references. I have also benefited from discussions with Piet Hut and Joe Traub, which went however a somewhat different direction [5].

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