Problemen

| Problem Section

Edition 2023-3 We received solutions from Rik Biel, Chris A. J. Klaassen, Thijmen Krebs, Lucía L. Pacios, Ana Pose, Andrés Ventas and Jan de Vries.

Problem 2023-3/A

Let $n>0$ be an integer and let $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ be an isometry, i.e., a map such that for all $x,y\in\mathbb{R}^n$ we have $|\varphi(x)-\varphi(y)|=|x-y|$. Let $X\subset\mathbb{R}^n$ be a set such that $\{\varphi(x)\mid x\in X\}\subseteq X$. Show that if X is closed and bounded, then $\{\varphi(x)\mid x\in X\}=X$, and show that we can drop neither of these two assumptions.

Solution This problem was solved in a collaboration by Lucía L. Pacios and Ana Pose and Andrés Ventas. Moreover, it was solved by Rik Biel and Thijmen Krebs. Jan de Vries has proved a more general result: an isometry on a compact metric space is surjective.

Let $y\in X$. By continuity of φ it suffices to prove that for every $\varepsilon>0$ there exists $x\in X$ such that $|\varphi(x)-y|<\varepsilon$. By Bolzano–Weierstrass, there exist $i>j$ such that $\left|\varphi^i(y)-\varphi^j(y)\right|<\varepsilon$. Using that φ is an isometry, we then find that $\left|\varphi^{i-j}(y)-y\right|<\varepsilon$. Now $x=\varphi^{i-j-1}(y)$ suffices.

Consider $X=\mathbb{Z}_{\geq 0}\subseteq\mathbb{R}$. Note that X is closed but not bounded, and that $x\mapsto x+1$ is an isometry that maps X to $X\backslash\{0\}$.

Identify $\mathbb{R}^2$ with $\mathbb{C}$ and take $X=\{\exp(n\mathrm{i})\mid n\in\mathbb{Z}_{\geq 0}\}$. Note that X is bounded but not closed, and that $x\mapsto\exp(\mathrm{i})\cdot x$ is an isometry that maps X to $X\backslash\{1\}$.

Problem 2023-3/B

Let X be a normally distributed random variable and let $t\in\mathbb{R}_{>0}$. Show that $x\mapsto\mathbb{P}(X\leq x+t\mid x\leq X)$ is an increasing function.

Solution This problem contained an error. Instead of 'increasing', it was stated 'decreasing'. A solution for the problem was sent in by Thijmen Krebs. A more general result was proven by Chris A. J. Klaassen: instead of for normal distributed random variables, it has been proved for random variables with a strongly unimodel distribution.

First note that without loss of generality we may assume that X follows a standard normal distribution, and write f and F for its probability density and cumulative density function, respectively. It suffices to show that $G(x)=F(x+t)/F(x)$ is a decreasing function, or equivalently, that

$$(\log G(x))'=\frac{f(x+t)F(x)-f(x)F(x+t)}{F(x)F(x+t)}<0$$

for all x. Hence it suffices to show for all x that

$$H(x)=e^{-tx-\frac{1}{2}t^2}F(x)-F(x+t)<0.$$

We claim that

$$\lim_{x\to-\infty}e^{-tx}F(x)=0.$$

Clearly 0 is a lower bound. Let $\delta>0$. Then there exists some $B<0$ such that $e^{-\frac{1}{2}x^2}\leq\delta e^{tx}$ for all $x<B$. Hence

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}z^2}\mathrm{d}z\leq\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\delta e^{tz}\mathrm{d}z=\frac{\delta}{\sqrt{2\pi}}\cdot e^{tx}$$

for such x, so $e^{-tx}F(x)\leq\frac{\delta}{\sqrt{2\pi}}$ for x sufficiently small. Taking $\delta\to 0$ we obtain the limit.

It follows from the claim that $\lim_{x\to-\infty}H(x)=0$. Since also

$$H'(x)=-te^{-tx-\frac{1}{2}t^2}F(x)+e^{-tx-\frac{1}{2}t^2}f(x)-f(x+t)=-te^{-tx-\frac{1}{2}t^2}F(x)<0,$$

we conclude that $H(x)<0$ for all x.

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Oplossingen | Solutions

Problem 2023-3/C (proposed by Hendrik Lenstra)

Let $p=2n+1$ be an odd prime and consider the finitely presented group G with generators $x_1,\dots,x_n$ and for each $0<i,j,k\le n$ such that $ij=k$ or $ij=p-k$ the relation $x_ix_j=x_k$. Show that G is a cyclic group of order n.

Solution A partial solution for this problem was given by Andrés Ventas.

We prove by induction on i the following statement:

$$\forall j,k\in\{1,\dots,n\},\ \ (ij\equiv\pm k \bmod p \Rightarrow x_ix_j=x_jx_i=x_k).$$

This is trivial for $i=1$. Now fix an integer $1<i\le n$ and assume that the statement holds for smaller values of i. We prove the statement for i. Let $j,k\in\{1,\dots,n\}$ be such that $ij\equiv\pm k \bmod p$. Consider the i sets $\{hj,hj+1,\dots,hj+\lfloor p/i\rfloor\}$ for $h=0,\dots,i-1$. Each of these sets contains $\lfloor p/i\rfloor+1$ elements, and as $i\cdot(\lfloor p/i\rfloor+1)>p$, we find that the sets are not pairwise disjoint modulo p. It follows that there exist integers $0\le h_1<h_2<i$ and an integer $0\le\varepsilon<p/i$ for which $h_2j\equiv h_1j\pm\varepsilon \bmod p$, and $h=h_2-h_1$ then yields $hj\equiv\pm\varepsilon \bmod p$. Note that $\varepsilon\ne 0$ since p is prime. By assumption we have $x_ix_\varepsilon=x_{i\varepsilon}$ and by the induction hypothesis applied to h we know that $x_hx_i=x_{ih}=x_ix_h$ and that $x_hx_k=x_{i\varepsilon}=x_kx_h$. This gives the following equalities

$$x_ix_jx_h=x_ix_\varepsilon=x_{i\varepsilon}=x_kx_h,$$
$$x_hx_jx_i=x_\varepsilon x_i=x_{\varepsilon i}=x_hx_k,$$

and thus $x_ix_j=x_k=x_jx_i$ which completes the induction step. Now it follows that $G\to\mathbf{F}_p^*/\{\pm1\}$ given by $x_i\mapsto i$ for all $i\in\{1,\dots,n\}$ is an isomorphism. Since $\mathbf{F}_p^*$ is cyclic of order $2n$, it follows that G is cyclic of order n.