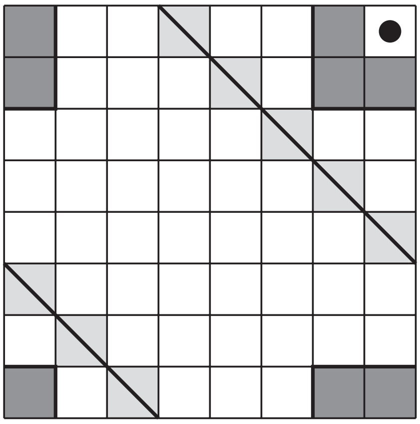
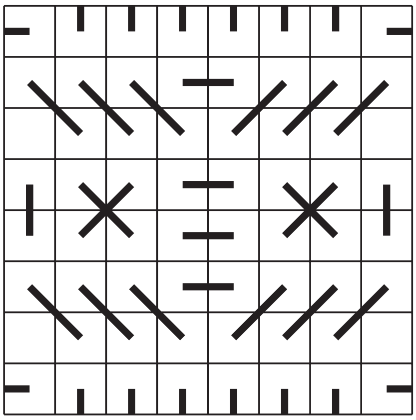
Edition 2021-3 We received solutions from Rik Biel, Mike Daas, Nicky Hekster, Thijmen Krebs and Wim Nuij.

Problem 2021-3/A (proposed by Daan van Gent)

Write $T=(\mathbb{Z}/8\mathbb{Z})^2$ for the *torus chessboard*. For every square $t\in T$ its *neighbours* are the squares in the set $\{t+d \mid d\in D\}$ for $D=\{-1,0,1\}^2\setminus\{(0,0)\}$. A *line* is a set of squares of the form $\{t+nd \mid n\in\mathbb{Z}\}$ for $t\in T$ and $d\in D$. The left figure on the next page gives an example of the neighbours (coloured dark grey) of the dotted square, as well as an example of a line (coloured light grey). Disprove or give an example: There exists a pairing of neighbouring squares, i.e. a partitioning of T into sets $\{s,t\}$ of size 2 such that s and t are neighbours, such that every line contains a pair.

Solution The right figure shows the solution by Thijmen Krebs (also solved by Wim Nuij):





Problem 2021-3/B (proposed by Hendrik Lenstra)

Write φ for the Euler totient function. Determine all infinite sequences $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ of positive integers satisfying $\varphi(a_{n+1})=a_n$ for all $n\geq 0$.

Solution Let $(a_n)_{n\geq 0}$ satisfy $\varphi(a_{n+1})=a_n$ for all $n\geq 0$. Write $X=\{2^a\cdot 3^b \mid a,b\in\mathbb{Z}_{\geq 0}\}$.
Claim: $(a_n)_n\subset X$.
Proof: We will prove this claim by contradiction, so assume that there exists an integer $N\geq 0$ for which $a_N\notin X$. Then for all $n\geq N$ we have that $a_n\notin X$, and without loss of generality $N=0$. Then the sequence $(\mathrm{ord}_2(a_n))_{n\geq 0}$ is weakly decreasing. Therefore, without loss of generality we may assume that the sequence $(\mathrm{ord}_2(a_n))_{n\geq 0}$ is constant. Now by dividing the sequence $(a_n)_n$ by $2^{\mathrm{ord}(a_0)-1}$ we may assume that $\mathrm{ord}_2(a_n)=1$ for all n. For every n, write $a_n=2b_n$ with $b_n>1$ odd. Then for every $n\geq 1$ we have that b_n is a prime power: otherwise $\mathrm{ord}_2(a_{n-1})=\mathrm{ord}_2(\varphi(b_n))$ would be too large. Hence write $b_n=p_n^{c_n}$ with p_n an odd prime and $c_n\in\mathbb{Z}_{\geq 1}$. Then for every $n\geq 2$ we see that $\varphi(b_n)/2=p_n^{c_n-1}\cdot(p_n-1)/2$ is a prime power, and since $p_n\neq 3$ we find that $c_n=1$. Hence, $(b_n)_{n\geq 2}$ is a sequence of prime numbers satisfying $b_n=2b_{n-1}+1$. Then for all $n\geq 0$ we find that $b_{n+2}=2^n(b_2+1)-1$. Let p be any prime divisor of b_2. Then p divides b_{n+2} whenever $2^n\equiv 1$ modulo p, so in this case $b_{n+2}=p$. This happens infinitely many times and is a contradiction since $(b_n)_n$ is strictly increasing. □

By the previous claim we conclude that for all $n\geq 0$ we have $a_n\in X$. The sequences $(a_n)_n\subset X$ that suffice the relation $\varphi(a_{n+1})=a_n$ for all $n\geq 0$ are precisely those $(a_n)_n$ for which $(a_{n+1}/a_n)_n$ is a weakly increasing sequence in $\{2,3\}$.

Problem 2021-3/C (proposed by Hendrik Lenstra)

Let R be a ring. We say $x\in R$ is a *unit* if there exists some $y\in R$ such that $xy=yx=1$, and $x\in R$ is *idempotent* if $x^2=x$. Show that if every unit of R is central, then every idempotent of R is central.

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Solution This solution is submitted by Mike Daas (also solved by Rik Biel and Nicky Hekster). Assume that every unit of R is central. Note that for every idempotent $e \in R$ it holds that $e(1-e) = (1-e)e = 0$.

Claim: $e \in R$ is a central idempotent if and only if $(1-e)Re = eR(1-e) = 0$.

Proof. If e is a central idempotent, then $(1-e)Re = (1-e)eR = 0$ and similarly $eR(1-e) = 0$. Now suppose $(1-e)Re = eR(1-e) = 0$. For any $x \in R$ we have $0 = (1-e)xe = xe - exe$ so $xe = exe$ and similarly $ex = exe$. Hence $xe = ex$ and e is central. Taking $x = 1$ in the above shows $e = e^2$, so e is idempotent. □

Let $e \in R$ be idempotent. Suppose $x \in (1-e)Re$. Then $x^2 = 0$, so $1+x$ has inverse $1-x$. By assumption $1+x$ is central. As 1 is trivially central we conclude that x is central. Then $x = (1-e)xe = (1-e)ex = 0$, so $(1-e)Re = 0$. Similarly $eR(1-e) = 0$, so e is central by the above.