Problemen

| Problem Section

Edition 2021-2 We received solutions from Rik Biel, Rob Eggermont and Sander Scholtus. The solution of Problem B will appear in a future issue.

Problem 2021-2/A (proposed by Onno Berrevoets)

A hot frying pan contains 2^{2021} potato slices. Each time we toss the slices, each slice has a chance of 0.5 to land on its other side. These probabilities are individually independent. How often do we need to toss the slices so that with probability at least 0.5 all slices will have lain on both sides?

Solution This solution is submitted by Sander Scholtus (also solved by Rik Biel). The answer is: 2022 times. Denote the probability that a potato slice has landed on both sides after tossing k times by p_k. It is clear that $p_0 = 0$ and $p_1 = \frac{1}{2}$. Generally, p_k equals 1 minus the probability that a potato slice has not flipped after tossing k times, i.e., we have $p_k = 1-(1-p_1)^k = 1-2^{-k}$ for all $k \in \{0,1,2,\dots\}$. By independence of the potato slices, we now find that all $N := 2^{2021}$ slices are flipped after tossing k times equals $p_k^N = (1-2^{-k})^N$. Hence, we are asked to find the minimal value of k for which we have

$$\left(1-\frac{1}{2^k}\right)^N > \frac{1}{2}.$$

The left hand side is clearly a strictly increasing function of k. Now consider $k = 2021$. In this case, we have

$$\left(1-\frac{1}{2^k}\right)^N = \left(1-\frac{1}{N}\right)^N.$$

As N is very large, we can approximate this number sufficiently well with $1/e < \frac{1}{2}$ and conclude that $k = 2021$ is too small to achieve our desired result. We try the next value $k = 2022$. In this case we find

$$\left(1-\frac{1}{2^k}\right)^N = \left(1-\frac{1}{2N}\right)^N = \sqrt{\left(1-\frac{1}{N}\right)^N} \approx \sqrt{\frac{1}{e}} > \frac{1}{2}.$$

Hence $k = 2022$ is the desired number to toss the frying pan.

Justifications of the approximations in the previous argument can be found in the following inequalities:

$$\left(1-\frac{1}{x}\right)^x < \frac{1}{e} < \left(1-\frac{1}{x}\right)^{x-1}.$$

Indeed, in the limit $x \to \infty$, these inequalities become equalities. Moreover, $(1-\frac{1}{x})^x$ and $(1-\frac{1}{x})^{x-1}$ are respectively increasing and decreasing functions, which can be shown by taking the derivative and using the Mercator series of the logarithm.

Problem 2021-2/C (proposed by Onno Berrevoets)

Let R be a commutative ring, and consider the set X of R-ideals J with $J^2 \neq J$. Suppose that I is a maximal element of X (with respect to inclusion). Prove that I is a maximal ideal of R.

Solution This solution is submitted by Rob Eggermont. Without loss of generality, we may assume $I^2 = 0$. To prove I is maximal, it suffices to prove that for all $x \in R \setminus I$, x is invertible $(\bmod I)$. Let $x \in R \setminus I$. By assumption of maximality, $(xR+I)^2 = xR+I$, so there are $r \in R$, $i \in I$ such that $x = rx^2 + ix$. Our aim is to show $rx = 1 (\bmod I)$. Write $u = 1-rx$. Note that we may rewrite $x = rx^2+ix$ as $xu = ix$. We claim that $u^3 = u^2$. Indeed, we have

$$u^3 = u^2(1-rx) = u^2 - rxu^2 = u^2 - rxi^2 = u^2.$$

from which it follows that $u^2(1-u) = 0$. As a consequence, we see that $(1-u)R \cap u^2R = 0$: for any $r,s \in R$ with $(1-u)s = u^2r$, we have $u^2r = u^4r = u^2(1-u)s = 0$.

Redactie: Onno Berrevoets en Daan van Gent
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

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Consider the ideal J generated by u^2 and I. Observe that $J^2 = (u^2R+I)^2 = u^4R + u^2I = u^2R$. If $u^2 \notin I$, then $J^2 = J$ by our assumption on maximality, so then J equals u^2R, hence $I \subseteq u^2R$. In particular, we have $(1-u)I = 0$ since $(1-u)u^2 = 0$. Consider the ideal J' generated by $(1-u)$ and I. We have $J'^2 \subseteq (1-u)R$. Recall that $(1-u)R \cap u^2R = 0$. Since I is not zero and contained in u^2R, it follows that J'^2 is strictly contained in J', hence $J' = I$, so $1-u \in I \subseteq u^2R$. From $(1-u)R \cap u^2R = 0$ it now follows that $1-u = 0$, so $u = 1$. However, this means $rx = 0$, and we find $x = ix$, contradicting our assumption that x is not in I. So we must have $u^2 \in I$. This yields $u^2 = u^4 = 0$.

It is easy to see that $(uR+I)^3 = 0$, since $I^2 = 0$ and $u^2 = 0$. Since $uR+I$ is not zero, it follows that $(uR+I)^2$ is strictly contained in $uR+I$. Hence $u \in I$, and therefore $rx = 1 (\mathrm{mod}\ I)$. So x is invertible $(\mathrm{mod}\ I)$, as was to be shown.