Problemen

| Problem Section

Edition 2021-1 We received solutions from Rik Bos, Pieter de Groen, Nicky Hekster, Marnix Klooster and Rutger Wessels.

Problem 2021-1/A (proposed by Daan van Gent and Hendrik Lenstra)

Let G and A be groups, where G is denoted multiplicatively and where A is abelian and denoted additively. Assume that A is 2-torsion-free, i.e. it contains no element of order 2.

Suppose that $q: G \to A$ is a map satisfying the parallelogram identity: for all $x, y \in G$ we have

$$q(xy) + q(xy^{-1}) = 2q(x) + 2q(y). \tag{1}$$

Prove that for all $x, y \in G$ we have $q(xyx^{-1}y^{-1}) = 0$.

Solution We received solutions from Rik Bos, Nicky Hekster, Marnix Klooster, and Rutger Wessels. The solution provided uses the one by Nicky.

We let 1 denote the identity element of G, and 0 the identity element of A. Substituting $y = 1$ yields $2q(1) = 0$, whence $q(1) = 0$ since A is **2**-torsion free. Substituting $x = 1$ in (1) gives

$$q(y) = q(y^{-1}) \quad \text{for all } y \in G. \tag{2}$$

In the case $x = y$, formula (1) implies

$$q(x^2) = 4q(x) \quad \text{for all } x \in G. \tag{3}$$

By switching the x and y in (1) we obtain $q(yx) + q(yx^{-1}) = 2q(y) + 2q(x)$, so using (2) we get $q(xy) + q(xy^{-1}) = q(xy) + q((yx^{-1})^{-1}) = q(xy) + q(yx^{-1}) = q(yx) + q(yx^{-1})$, hence

$$q(xy) = q(yx) \quad \text{for all } x, y \in G. \tag{4}$$

Applying (1) with x replaced by xy we see that

$$q(xy^2) = 2q(xy) - q(x) + 2q(y) \quad \text{for all } x, y \in G. \tag{5}$$

Next we claim that

$$q(xy^2x) = 4q(xy) \quad \text{for all } x, y \in G. \tag{6}$$

Indeed,

$$\begin{aligned} q(xy^2x) &\underset{(4)}{=} q(x^2y^2) \underset{(5)}{=} 2q(x^2y) - q(x^2) + 2q(y) \underset{(4)(3)}{=} 2q(yx^2) - 4q(x) + 2q(y) \\ &\underset{(5)}{=} 2(2q(yx) - q(y) + 2q(x)) - 4q(x) + 2q(y) \underset{(4)}{=} 4q(xy). \end{aligned}$$

It follows that

$$q(xyx^{-1}y^{-1}) = q((xy)(yx)^{-1}) \underset{(1)}{=} 2q(xy) + 2q(yx) - q(xy^2x) \underset{(4)(6)}{=} 4q(xy) - 4q(xy) = 0.$$

Problem 2021-1/B (folklore)

Prove that every Jordan curve (i.e. every non-self-intersecting continuous loop in the plane) contains four points A, B, C, D such that $ABCD$ forms a rhombus.

Solution We received a partial solution from Pieter de Groen. This is a sketch of a solution that is largely based on that, and will work in many simple cases. A full proof by Mark Nielsen can be found at https://link.springer.com/content/pdf/10.1007/BF01265340.pdf. Even more generally, Arthur Milgram proved in 1939 (published in PNAS in 1943 as 'Some Topologically Invariant Metric Properties') that for all $n \geq 3$, every Jordan curve contains an inscribed equilateral n-gon. It remains an open problem whether any Jordan curve contains an inscribed square. This problem is known by several names, such as the square peg problem, the inscribed square problem, or the Toeplitz' conjecture.

Redactie: Onno Berrevoets, Rob Eggermont en Daan van Gent

problems@nieuwarchief.nl

www.nieuwarchief.nl/problems

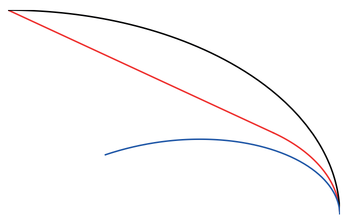
Let J be a Jordan curve, i.e. a non-self-intersecting continuous loop $J:[0,1]\to\mathbb{R}^2$. We write $J(t)=(x(t),y(t))$. Without loss of generality, we may assume that $J(0)=(0,0)=J(1)$. Let $|J(t)|=\sqrt{x(t)^2+y(t)^2}$ be the distance from $J(t)$ to the origin. Let m be the absolute maximum of $|J(t)|$, and let $t_m\in(0,1)$ such that $|J(t_m)|=m$. Since J is continuous, for every $0<r<m$, the equation $|J(t)|=r$ has a solution on both $(0,t_m)$ and $(t_m,1)$. Let $S(r):=\{J(t)\mid |J(t)|=r, t\in[0,t_m]\}$ and let $S'(r):=\{J(t)\mid |J(t)|=r, t\in[t_m,1]\}$. The union S of the sets $S(r)$ forms a closed, simply connected set (this being the image of the restriction of J to $[0,t_m]$), and so does the union S' of the sets in $S'(r)$.

Given the origin $A:=(0,0)$ and two distinct points $B=J(s_1)$, $C=J(s_2)$ at equal distance from the origin, the points A, B, C and $D:=B+C$ are the vertices of a rhombus. In particular, for any $r\in(0,m]$, any pair of elements $B\in S(r)$ and $C\in S'(r)$defines a rhombus with vertices A, B, C on J and top-vertex $D=B+C$.

We let $T(r)=\{B+C\mid B\in S(r), C\in S'(r)\}$ and $T=\bigcup_{r\in(0,m]}T(r)$. The set T is connected. Namely, the union of the sets $S(r)\times S'(r)\subset\mathbb{R}^4$ is connected, and T is the image of a continuous map with the former union as its domain. Note that there are some subtleties involved in showing connectedness of the union of the sets $S(r)\times S'(r)$; if one wants to walk from (B,C) to (B',C'), we cannot simply walk from B to B' and take the same path from C to C', since for example on one path, the distance to the origin may be increasing while on another path, it might sometimes decrease. One may need to backtrack on occasion. We will not give a full proof here (and there may be an issue with sufficiently complicated Jordan curves).

We show that T contains a point in the exterior of J. Consider the curve near $(x(t_m),y(t_m))$. Then either there is a closed interval $[a,b]$ with $a\neq b$ containing t_m such that for all $a\le t\le b$ the distance from $(x(t),y(t))$ to $(0,0)$ is equal to m, or no such interval exists. In both cases, for all $\epsilon>0$, there is some $r\le r_m$ such that there exist $t_1\in S(r), t_2\in S'(r)$ with $t_1\neq t_2$ such that the distance from $(x(t_i),y(t_i))$ to $(x(t_m),y(t_m))$ is less than ϵ. For ϵ sufficiently small, letting $B=(x(t_1),y(t_1))$ and $C=(x(t_2),y(t_2))$ yields $|D|>m$, so D lies in the exterior of the curve. If we can also find an element of T that is in the interior of the curve, we will be done, since by connectedness, T must intersect the curve somewhere.

The problem is that this does not necessarily happen; here is an example in which this is not the case:



The Jordan curve is the union of the top black path and the middle red path. Any rhombus with three vertices on the black path $(\cos(t),\sin(t))$ for $0\le t\le\pi/2$ including the point $(1,0)$ will have its fourth vertex on the bottom blue path, parameterized by $(\cos(2t)-(\cos(t)-1),$ $\sin(2t)-\sin(t))$ for $0\le t\le\pi/4$. The red path only intersects the blue path at $(1,0)$. The round part of the red path is part of the circle centered at $(0.5,0)$ with radius 0.5.

To fix this, we may need to choose a different starting point. Note that our argument that T contained a point outside of the curve was independent of our starting point A, so it now suffices to find any rhombus with three points on J such that its fourth point lies in the interior of J. By the Jordan curve theorem, the interior of J (is a well-defined concept and) is non-empty. Pick any point P in the interior. Since J is compact, there is a point B' at minimal positive distance from P. The line passing through BP intersects J in some other point C' such that P lies on the line segment $B'C'$. Again by compactness, we may assume the line segment $B'C'$ does not intersect J in any points other than B' and C'. Without loss of generality, P is the midpoint of this line segment. The perpendicular bisector of $B'C'$ intersects J again; let A be a point of this intersection at minimal distance to P. If there are two such points, we have found a rhombus. Otherwise, by choosing A to be the origin, we find $D'=B'+C'$ lies in the interior of J (since otherwise, there would be a point on the bisector closer to P that is also in J). The previous part of the argument now shows that the set T contains a point D that lies on J. This concludes the proof.

Oplossingen | Solutions

Problem C (proposed by Daan van Gent)

A *directed binary graph* is a finite vertex set V together with maps $e_1, e_2 : V \to V$. (The edges are formed by the ordered pairs $(v, e_i(v))$ with $i \in \{1,2\}$.)

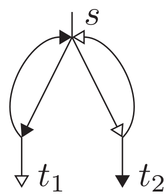
For $a,b,c,d \in \mathbb{Z}_{>0}$, an *$(a:b)$-to-$(c:d)$ distributive graph* is a directed binary graph G together with distinct vertices $s, t_1, t_2 \in V$ such that G interpreted as a Markov chain has the following properties:

1. For all $v \in V$ the edges $(v, e_1(v))$ have transition probability $\frac{a}{a+b}$ and edges $(v, e_2(v))$ have probability $\frac{b}{a+b}$.
2. It has the initial state s with probability 1.
3. Both t_1 and t_2 connect to themselves, meaning $e_i(t_j) = t_j$ for all $i,j \in \{1,2\}$.
4. It has a unique stationary distribution of t_1 with probability $\frac{c}{c+d}$ and t_2 with probability $\frac{d}{c+d}$.

Show that for all $a,b,c,d \in \mathbb{Z}_{>0}$ there exists an $(a:b)$-to-$(c:d)$ distributive graph.

Solution We received no submissions from readers. This solution was provided by Daan van Gent.

First we construct $(a:b)$-to-$(1:1)$ distributive graph for all a and b:



By 'substitution' it suffices to construct a $(1:1)$-to-$(c:d)$ distributive graph for all c and d. Let $k \in \mathbb{Z}_{\geq 0}$ such that $c + d \leq 2^k$. Make a complete binary tree with root s and such that there are 2^k leaves of equal distance to s. Connect c of those to t_1 and d of those to t_2 and connect the remaining $2^k - c - d$ to s.