Problemen

| Problem Section

Edition 2018-1 We received solutions from Mohamed Aasila (Strasbourg), Yagub Aliyev (Baku), Bitan Basu (Kolkata), Pieter de Groen (Brussel), Alex Heinis (Amsterdam), Hans van Luipen (Zaltbommel), Abel Putnoki (Eindhoven), Hendrik Reuvers (Maastricht), Hans Samuels Brusse (Den Haag), Seb Schilt (Den Haag), B. Sury (Bengaluru), Araz Yusubov (Baku) and Rob van der Waall (Huizen). The book tokens go to Abel Putnoki, Seb Schilt and Yagub Aliyev.

Problem 2018-1/A

Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6))=4$ and $f(f(25))=3$. Prove that if p is prime then $f(p)$ is also prime.

Solution Solved by Mohamed Aasila, Yagub Aliyev, Bitan Basu, Pieter de Groen, Alex Heinis, Hans van Luipen, Abel Putnoki, Hendrik Reuvers, Hans Samuels Brusse, B. Sury, Rob van der Waall.
We follow Abel Putnoki's solution. A prime has two divisors, i.e., $f(f(p))=2$. Applying f once more gives $f(f(f(p)))=f(2)$. Hence, the number of divisors of $f(p)$ is equal to $f(2)$. We need to show that $f(2)=2$.
Since 2 is prime, the number of divisors of $f(2)$ is equal to $f(2)$. The only numbers with this property are 1 and 2. Therefore $f(2)$ is either equal to 1 or 2. We need to rule out 1. Arguing by contradiction, suppose that $f(2)=1$. Then $f(p)$ has only one divisor, which implies $f(p)=1$. Applying f once more to $f(2)=1$ gives $f(1)=f(f(2))=2$. Now $f(f(4))=3$ is prime and therefore $f(f(f(4)))=1$. In other words, $f(4)$ has only one divisor, which implies that $f(4)=1$. This is absurd because if we apply f once more we get $f(1)=f(f(4))=3$, contradicting that $f(1)=2$.
We conclude that the set of primes is invariant under f if $f^2(n)$ is equal to the number of divisors of n. This still leaves us with the question whether such an f exists.

Problem 2018-1/B

Let n be a positive integer and $F\subseteq 2^{[n]}$ a family of subsets of $[n]=\{1,2,\dots,n\}$ that is closed under taking intersections. Suppose that
1. For every $A\in F$ we have: $|A|$ is not divisible by 3.
2. For every pair $i,j\in[n]$ there is an $A\in F$ such that $i,j\in A$.
Show that n is not divisible by 3.

Solution This problem was proposed by Dion Gijswijt, who took it from a recent paper by Martin Nägele, Benny Sudakov and Rico Zenklusen, 'Submodular minimization under congruency constraints', arXiv 1707.06212v2. We received solutions from Hendrik Reuvers, Hans Samuels Brusse and Seb Schilt. Below is the solution by Seb Schilt.
First consider what happens if each $|A|$ is *divisible* by 3. By inclusion-exclusion

$$|\bigcup F|=\sum_{A\in F}|A|-\sum_{A,A'\in F}|A\cap A'|+\sum_{A,A',A''\in F}|A\cap A'\cap A'|-\cdots$$

All cardinalities are divisible by 3 since F is intersection-closed, so we have that n is divisible by 3. We only need here that every *element* is in some A, rather than every *pair*. The same idea works if all $|A|$ are 2 modulo 3. Simply add a dummy element to $[n]$ and to all A. If $[n]=\bigcup F$ for an intersection-closed F such that all $|A|\equiv j \bmod k$, then $n\equiv j \bmod k$.
We can reduce our problem to this situation by being clever. Let B^2 denote the set of ordered pairs $\{(b_1,b_2);b_1,b_2\in B\}$. The family F^2 of all A^2 is intersection-closed and contains all elements of $[n]^2$. Now $|A^2|=|A|^2\equiv 1 \bmod 3$. Therefore, $n^2\equiv 1 \bmod 3$. In other words, n is not divisible by 3.
What happens if we replace 3 by m in condition 1, and pair by $(m-1)$-tuple in condition 2? Nägele, Sudakov and Zenklusen show that the result remains true if m is a prime power, but it is false for arbitrary m.

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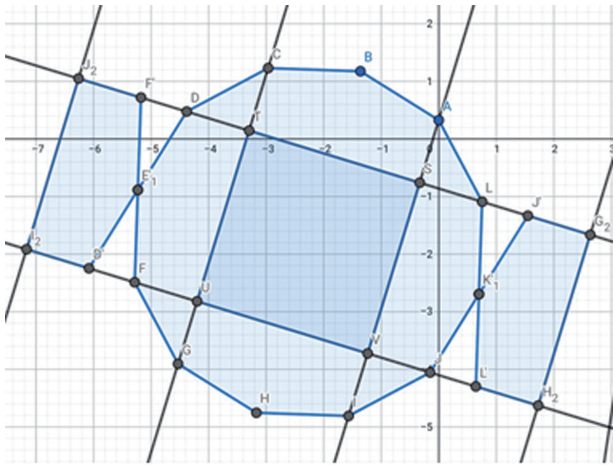
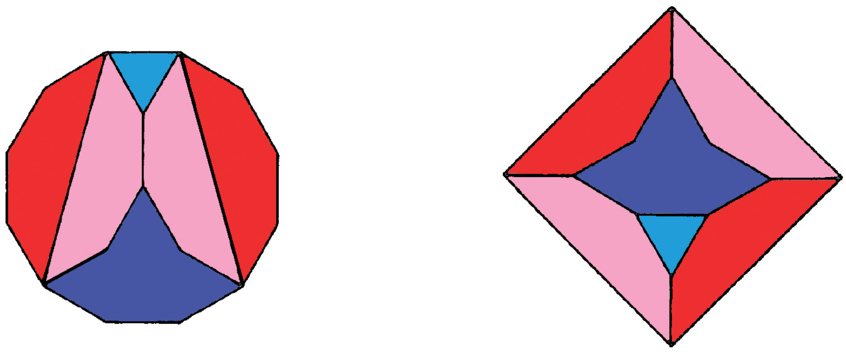
Problem 2018-1/C (proposed by Hendrik Lenstra)

Cut three squares of equal size in exactly the same way into three pieces each in such a way that the resulting nine pieces can be rearranged to form a regular twelve-gon. Open question: Can you cut the three squares into *eight* pieces that form a regular twelve-gon?

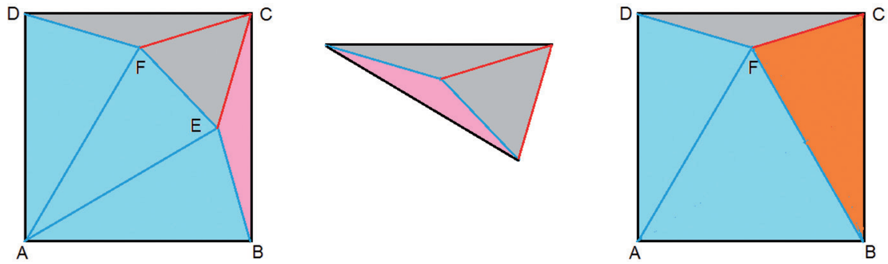
Solution Solved by Yagub Aliyev and Araz Yusubov, Hendrik Reuvers, Hans Samuels Brusse. Yagub Aliyev writes that the entire computer science department of the ADA University in Azerbaijan got interested in this problem and everybody worked on it. Ranging from the secretaries all the way down to the dean. Apart from solving the nine-piece puzzle, they came up with several other interesting ways to divide the twelve-gon into nine pieces that almost assemble into three squares. One of these is illustrated below:



Hendrik Reuvers sends us a dissection of the dodecagon, aka the twelve-gon, from Harry Lindgren's delightful *Recreational Problems in Geometric Dissections*, Dover, 1972. On page 41 it says: "...the chord subtending four sides of a dodecagon is equal to the side of the equivalent square. Knowing this, we can find several dissections merely by trial on the dodecagon, and in almost embarrassing abundance." As illustrated by:



We now follow Hans Samuels Brusse's solution. First observe that the regular twelve-gon has area 3. Therefore, the three squares are unit squares. Think of the twelve-gon as a pie which has twelve triangular pieces. Let $ABCD$ be a square containing one quarter of the twelve-gon with A the circumcentre, B and D are vertices of the twelve-gon. Let E and F be the two intermediate vertices between B and D. Now observe that ABF is equilateral and the isosceles triangles DFC and BEF are congruent. We cut out the quarter of the twelve-gon $ABEFD$ and the remaining part along CF. If we now lay DFC on top of BEF, the two remaining pieces put together form one additional piece of pie. Repeating this for the other two squares gives two additional pieces. These three pieces piece together to form the remaining quarter of the twelve-gon. Piece of cake!



There is a second way to cut the pie. The pieces are $ABFD$, BCF and CDF. Observe that we can combine CDF and $ABFD$ to get $ABEFD$. The remaining piece CDF goes to the fourth quarter. This second solution is actually nicer, since now all pieces are convex.

The problem on eight-pieces-or-less remains open. Finding dissections is difficult. Proving that no dissection exists is even harder. Is it possible to prove that no dissection into eight *convex* pieces exists? Perhaps our friends at the Caspian See would like to try their hands at this.