Problemen

| Problem Section

Edition 2016-1 We received solutions from Raymond van Bommel, Rik Bos, Alexandros Efthymiadis, Pieter de Groen, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Toshihiro Shimizu, Djurre Tijsma and Martijn Weterings.

Problem 2016-1/A (proposed by folklore)

Let $a,b>1$ be integers such that $2a \le b$. Does there exist a map

$$f:\{a,a+1,\dots,b\}\to\{a,a+1,\dots,b\}$$

without fixed points, such that for all $n\in\{a,a+1,\dots,b\}$ we have $f^{f(n)}(n)=n$? Here, for a positive integer k, f^kdenotes the k-fold composition

$$\underbrace{f\circ f\circ\cdots\circ f}_{k\text{ times}}$$

of f.

Solution We received solutions from Raymond van Bommel, Rik Bos, Alexandros Efthymiadis, Pieter de Groen, Alex Heinis, Thijmen Krebs, Toshihiro Shimizu and Martijn Weterings. The book token is awarded to Rik Bos. Most of the solutions received are similar, and the following one is based on that.

The answer is no. Suppose for a contradiction that such an f does exist.

First note that f is surjective and therefore a permutation of $\{a,a+1,\dots,b\}$; write f as a product of disjoint cycles. Note that the cycle containing n is also the cycle containing $f^{-1}(n)$, and by $f^n(f^{-1}(n))=f^{f^{-1}(f(n))}(f^{-1}(n))=f^{-1}(n)$, we see that this cycle has length dividing n. It follows that the length of a cycle divides every element contained therein. Moreover, as f has no fixed points, every cycle has length greater than 1.

By Bertrand's postulate, there exists a prime p such that $\frac{1}{2}b<p<b$. Then note that the cycle containing p has length p, and that therefore every element in this cycle is divisible by p. Therefore this cycle contains an element that is at least $2p$ and therefore greater than b; this is the required contradiction.

Problem 2016-1/B (proposed by folklore)

Let $n\ge 3$ be an integer. Two players play the following game. Starting with a sheet of paper with the numbers 1 and 2 on them, the players take turns writing down a new number from 1 to n that is the sum of two numbers already on the sheet. The player who writes down the number n wins.

For which n does the first player have a winning strategy?

Solution We received solutions from Raymond van Bommel, Pieter de Groen, Alex Heinis and Thijmen Krebs. The book token is awarded to Thijmen Krebs. The following solution is based on that of Thijmen Krebs and that of Raymond van Bommel.

Consider the equivalent game where the first player starts by writing 1 on an empty sheet. We say a number $1\le m<n$ is *safe* if m and $n-m$ are distinct and not on the sheet. Note that the first player who writes down an unsafe number loses, as the next player can write down the number n in his next turn. We show that the first player wins if and only if $n\equiv 3,4 \bmod 4$.

The strategy for the winning player is simply to write down the smallest safe number, if it exists. This is always possible, as we show below.

Suppose that at any point, the number m is the smallest safe number. If $m=1,2$, then it is clear that m can be written down. So suppose that $m\ge 3$. Then at least $m-1$ turns have passed, so there are at least $m-1$ numbers on the sheet, and as the winning player has written down the smallest safe number in each of his previous turns, he has written down at least $\frac{1}{2}(m-1)$ numbers that are at most $m-1$. Moreover, as $m\ge 3$, the losing player has written down at least one number that is at most $m-1$, namely 1 or 2. Therefore there are at least $\frac{1}{2}(m+1)$ numbers on the sheet that are at most $m-1$. By the pigeonhole principle, among these there are at least 2 (possibly the same) that add up to m, so m can be written down.

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Now note that with this strategy, the number of safe numbers decreases by 2 every turn (or the other player loses immediately). As the number of safe numbers in the beginning is $2\lfloor \frac{1}{2}(n-1) \rfloor$, which is 2 modulo 4 if and only if $n \equiv 3,4 \bmod 4$, we find that the first player can win if and only if $n \equiv 3,4 \bmod 4$.

Problem 2016-3/C (proposed by Hendrik Lenstra)

Determine all two-sided infinite sequences of positive integers in which each number is the Euler-phi of the next.

Solution We received solutions from Raymond van Bommel, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Toshihiro Shimizu, Djurre Tijsma and Martijn Weterings. The book token is awarded to Raymond van Bommel, whose solution the following one is based on.
We will make use of the following. Let ϕ denote the Euler-phi function, and let $v_2(a)$ denote the number of prime factors 2 in the prime factorisation of a positive integer a.

Lemma. *Let a be a positive integer. If $\phi(a)$ contains a prime factor of at least 5, then so does a. Moreover, in this case we have $v_2(\phi(a)) \geq v_2(a)$, with equality if and only if a is of the form $2^e p^f$ with $p \geq 5$ prime congruent to 3 modulo 4.*

Proof. If a does not contain a prime factor of at least 5, then neither does $\phi(a)$. If $a = 2^e p_1^{f_1} \cdots p_s^{f_s}$ with $p_1, \ldots, p_s$ distinct odd primes and $f_1, \ldots, f_s > 0$, then $v_2(\phi(a)) = e + \sum_{i=1}^{s} v_2(p_i - 1) - 1$. As for all i we have $v_2(p_i - 1) > 0$, it follows that $v_2(\phi(a)) \geq v_2(a)$, with equality if and only if $s = 1$ and $p_1 \equiv 3 \bmod 4$. □
Now let $(a_n)_{n \in \mathbb{Z}}$ be a sequence in which each number is the Euler-phi of the next, so that $a_n = \phi(a_{n+1})$ for all $n \in \mathbb{Z}$. We first show that no a_n can have a prime factor of at least 5.
Suppose the contrary: let $N \in \mathbb{Z}$ be such that a_N has a prime factor of at least 5. By the lemma, we see that all for all $n \geq N$ the number a_n has a prime factor of at least 5, and that therefore the sequence $(v_2(a_n))_{n \geq N}$ is decreasing. Hence this sequence stabilises, and we assume (by replacing N by a larger number if necessary) that $(v_2(a_n))_{n \geq N}$ is constant.
We see that for all $n > N$, we have $a_n = 2^{e_n} p_n^{f_n}$ for some prime $p_n \geq 5$ congruent to 3 modulo 4, and $f_n > 0$. As

$$2^{e_n} p_n^{f_n} = a_n = \phi(a_{n+1}) = 2^{e_{n+1}-1} p_{n+1}^{f_{n+1}-1} (p_{n+1} - 1),$$

with $v_2(p_{n+1} - 1) = 1$ and $p_{n+1} - 1 \geq 4$, we see that $p_{n+1} - 1$ contains an odd prime factor, which therefore must be p_n by uniqueness of prime factorisation. As therefore $p_{n+1} > p_n$, it follows that $f_{n+1} = 1$ and that $p_{n+1} - 1 = 2p_n$.
Now note that the map $F: \mathbb{Z}/p_{N+1}\mathbb{Z} \to \mathbb{Z}/p_{N+1}\mathbb{Z}$, $x \mapsto 2x + 1$ is bijective as p_{N+1} is odd, and therefore has a finite order as permutation on $\mathbb{Z}/p_{N+1}\mathbb{Z}$. Therefore there exists an integer $M > 0$ such that $p_{N+1+M} \equiv 0 \bmod p_N$, which contradicts the fact that p_{N+1+M} is a prime greater than p_{N+1}. Hence no a_n can have a prime factor of at least 5.
Write therefore, for all $n \in \mathbb{Z}$, $a_n = 2^{e_n} 3^{f_n}$. Then $e_n > 0$ unless $a_n = 1$, since otherwise a_n is not the Euler-phi of any positive integer. We then have

$$a_n = \phi(a_{n+1}) = \begin{cases} 2^{e_{n+1}} 3^{f_{n+1}-1} & \text{if } e_{n+1}, f_{n+1} > 0; \\ 2^{e_{n+1}-1} & \text{if } e_{n+1} > 0 \text{ and } f_{n+1} = 0; \\ 1 & \text{if } e_{n+1} = f_{n+1} = 0. \end{cases}$$

Hence we see that (a_n) must have one of the following forms.

- If there exists an a_n containing a prime factor 3, then the sequence is of the form

  $$\ldots, 1, 1, 1, 2, \ldots, 2^{e-1}, 2^e, 2^e 3, 2^e 3^2, \ldots$$

  with $e > 0$.
- If no a_n contains a prime factor 3, but there exists one containing a prime factor 2, then the sequence is of the form

  $$\ldots, 1, 1, 1, 2, 2^2, \ldots$$

- Otherwise, the sequence is

  $$\ldots, 1, 1, 1, \ldots$$