Problemen

| Problem Section

Edition 2015-3 We received solutions from Raymond van Bommel, Josephine Buskes, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Guido Senden, Djurre Tijsma, Steven Troch, Traian Viteam and Robert van der Waall.

Problem 2015-3/A (folklore)

Determine

$$\sum_p \frac{1}{p} \prod_{q<p} (1 - \frac{1}{q}),$$

where p ranges over all prime numbers, and q ranges over all prime numbers less than p.

Solution We received solutions from Raymond van Bommel, Josephine Buskes, Alex Heinis, Thijmen Krebs, Hendrik Reuvers, Djurre Tijsma, Steven Troch, Traian Viteam and Robert van der Waall. The book token goes to Josephine Buskes, whose solution the following is based on.

In the following, sums and products always range over prime numbers. We rewrite the given series as a telescoping series:

$$\begin{aligned}\frac{1}{p} \prod_{q<p} (1 - \frac{1}{q}) &= \Big(1 - (1 - \frac{1}{p})\Big) \prod_{q<p} (1 - \frac{1}{q}) \\ &= \prod_{q<p} (1 - \frac{1}{q}) - \prod_{q \le p} (1 - \frac{1}{q}).\end{aligned}$$

As $\prod_p (1 - \frac{1}{p}) = 0$, it then follows that

$$\sum_p \frac{1}{p} \prod_{q<p} (1 - \frac{1}{q}) = 1.$$

Problem 2015-3/B (proposed by Wouter Zomervrucht)

Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of natural numbers, and let $\mathbb{N}^{2015}$ denote the set of 2015-tuples $(a(1), a(2), \ldots, a(2015))$ of natural numbers. We equip $\mathbb{N}^{2015}$ with the partial order $\preceq$ for which $a \preceq b$ if and only if $a(k) \le b(k)$ for all $k \in \{1, 2, \ldots, 2015\}$. We say that a sequence $a_1, a_2, \ldots$ in $\mathbb{N}^{2015}$ is good if for all $i < j$ we have $a_i \npreceq a_j$.

– Show that all good sequences are finite.

We say that a sequence $a_1, a_2, \ldots$ in $\mathbb{N}^{2015}$ is perfect if it is good and for all i and for all $k \in \{1, 2, \ldots, 2015\}$ we have $\{a_i(k) \le 2015i$.

– Does there exists a positive integer N such that all perfect sequences have length at most N?

Solution We received solutions from Raymond van Bommel, Alex Heinis and Guido Senden, and a solution of the first part from Djurre Tijsma. The book token goes to Alex Heinis, whose solution the following is based on.

For the first part, first note that every infinite sequence $x_1, x_2, \ldots$ of natural numbers has a non-decreasing subsequence; indeed, we can define it recursively by setting k_1 to be the minimal index for which x_{k_1} is minimal, and by setting k_{i+1} to be the minimal index larger than k_i for which $x_{k_{i+1}} = \min_j \{x_j : x_j \ge x_{k_i}\}$ for all i.

So suppose for a contradiction that there exists an infinite good sequence $a_1, a_2, \ldots$ in $\mathbb{N}^{2015}$. By passing to a subsequence at most 2015 times, we obtain an infinite good sequence $a'_1, a'_2, \ldots$ in $\mathbb{N}^{2015}$ such that $a'_i(k) \le a'_j(k)$ for all i, j, k, i.e. $a'_i \preceq a'_j$ for all i, j; a contradiction. Therefore no infinite good sequence can exist.

For the second part, suppose for a contradiction that for every positive integer N there exists a perfect sequence of length at least N. Since every good sequence is finite by the first part, this implies that there are infinitely many perfect sequences. Note that in any perfect sequence $a_1, a_2, \ldots$, the i-th term is an element of the finite set $\{1, 2, \ldots, 2015i\}^{2015}$.

By the pigeonhole principle, we can now recursively define a_i to be an element of

Redactie:

Gabriele Dalla Torre
Christophe Debry
Jinbi Jin
Marco Streng
Wouter Zomervrucht

Problemenrubriek NAW
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden

problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

$\{1,2,\ldots,2015i\}^{2015}$ such that there are infinitely many perfect sequences that start with $a_1, a_2, \ldots, a_i$. This defines an infinite sequence $a_1, a_2, \ldots$. This sequence is good; for all $i < j$ we note that $a_1, a_2, \ldots, a_j$ occurs as the first j terms of a perfect sequence, so $a_i \not\preceq a_j$. But this contradicts the non-existence of infinite good sequences obtained in the first part.

Problem 2015-3/C (proposed by Marcel Roggeband)

The (first) Bernoulli numbers B_n for integers $n \geq 0$ are defined by the following recursive formula.

$$B_0 = 1,$$
$$B_n = -\sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i}{n-i+1} \quad \text{for } n > 0.$$

Show that the Bernoulli numbers satisfy the following identity for all $n > 1$

$$B_n = n! \sum_{i=1}^{n} \sum_{\sigma} \frac{(-1)^{i-1}}{\sigma_1! \cdots \sigma_i!} \left(\frac{1}{2} - \frac{1}{\sigma_i + 1}\right)$$

In this sum, σ runs through all ordered i-tuples $(\sigma_1, \ldots, \sigma_i)$ of integers such that $\sigma_1 + \cdots + \sigma_i = n + i - 1$ and $\sigma_j \geq 2$ for all j.

Solution We received solutions from Raymond van Bommel, Alex Heinis, Thijmen Krebs and Hendrik Reuvers. The book token goes to Hendrik Reuvers, whose solution the following is based on.

Write $C_n = \frac{B_n}{n!}$, so that the C_k satisfy $C_0 = 1$ and

$$C_n = -\sum_{i=0}^{n-1} \frac{C_i}{(n-i+1)!}.$$

Moreover, write

$$A_n = \sum_{i=1}^{n} \sum_{\sigma} \frac{(-1)^{i-1}}{\sigma_1! \cdots \sigma_i!} \left(\frac{1}{2} - \frac{1}{\sigma_i + 1}\right)$$

where in the sum, σ runs through all ordered i-tuples $(\sigma_1, \ldots, \sigma_i)$ of integers such that $\sigma_1 + \cdots + \sigma_i = n + i - 1$ and $\sigma_j \geq 2$ for all j.

We are done once we show that $A_n = C_n$ for all $n \geq 2$. We do so by induction on n. For $n = 2$, we have $A_n = \frac{1}{12} = C_n$. So suppose that for some $m \geq 2$ we have $A_k = C_k$ for all k such that $2 \leq k \leq m$. We then show that $A_{m+1} = C_{m+1}$.

$$\begin{aligned} C_{m+1} &= -\sum_{k=0}^{m} \frac{C_k}{(m-k+2)!} = \frac{-1}{(m+2)!} + \frac{\frac{1}{2}}{(m+1)!} + \sum_{k=2}^{m} \frac{-A_k}{(m-k+2)!} \\ &= \frac{1}{(m+1)!}\left(\frac{1}{2} - \frac{1}{m+2}\right) + \sum_{k=2}^{m} \sum_{i=1}^{k} \sum_{\sigma} \frac{(-1)^i}{\sigma_1! \cdots \sigma_i!} \frac{1}{(m-k+2)!} \left(\frac{1}{2} - \frac{1}{\sigma_i + 1}\right), \end{aligned}$$

where in the sum, σ runs through all ordered i-tuples $(\sigma_1, \ldots, \sigma_i)$ of integers such that $\sigma_1 + \cdots + \sigma_i = k + i - 1$ and $\sigma_j \geq 2$ for all j.

Note that the first term in the expression obtained for C_{m+1} equals the first term of A_{m+1}. Moreover, for $i \geq 1$, the (k, i, σ)-term in the expression obtained for C_{m+1} equals the -term of A_{m+1}, where $\sigma' = (m-k+2, \sigma_1, \ldots, \sigma_i)$; this tuple is admissible since $(m-k+2) + (k+i-1) = (m+1) + (i+1) - 1$. The corresponding map $(k, i, \sigma) \mapsto (i+1, \sigma')$ on index sets is a bijection (with inverse given by $(j, \sigma') \mapsto (m+2-\sigma'_1, j-1, \sigma)$ where $\sigma = (\sigma'_2, \ldots, \sigma'_j)$), as desired.