Problemen

Problem Section

Edition 2014-4 We received solutions from Raymond van Bommel and Julian Lyczak, Alex Heinis, Alexander van Hoorn, Jos van Kan, Thijmen Krebs, Gerard Renardel, Hendrik Reuvers and Kees Vugs.

Problem 2014-4/A (proposed by Jan Turk)
Let $k > 3$ be an integer. Determine the variance of the greatest common divisor of k positive integers. Here we mean the limit, as $n \to \infty$, of the variance of the greatest common divisor of k integers in $\{1, 2, \ldots, n\}$ with respect to the uniform distribution on $\{1, \ldots, n\}^k$.

Solution We received solutions from Raymond van Bommel and Julian Lyczak, Alex Heinis, Alexander van Hoorn, Thijmen Krebs and Hendrik Reuvers. The book token goes to Alex Heinis. This solution is based on that of Thijmen Krebs.
We show that the desired variance is

$$\frac{\zeta(k-2)\zeta(k) - \zeta(k-1)^2}{\zeta(k)^2},$$

where ζ denotes the Riemann zeta function.
Let r be a positive integer, and let $E_{k,r}$ denote the function sending a positive integer n to the expectance of the r-th power of the greatest common divisor of k integers in $\{1, \ldots, n\}$. In other words, for all positive integers r and n, we have

$$n^k E_{k,r}(n) = \sum_{a \in \{1,\ldots,n\}^k} \gcd(a)^r.$$

Let J_r denote the Jordan totient function, i.e. the function sending i to the number of elements $j \in \{1, \ldots, i^r\}$ such that $\gcd(j, i^r)$ is r-th power free. We show that

$$n^k E_{k,r}(n) = \sum_{i=1}^{n} J_r(i) \lfloor \tfrac{n}{i} \rfloor^k.$$

We interpret the i-th term on the right hand side as counting the k-tuples divisible by i, each with a weight $J_r(i)$. Therefore on the right hand side, each k-tuple a is counted with a weight

$$\sum_{d \mid \gcd(a)} J_r(d).$$

So to prove our claim, it suffices to show that for all positive integers n, we have $\sum_{d \mid n} J_r(d) = n^r$. For this, note that for all positive integers m, n we have that $J_r(n)$ is also the number of elements $i \in \{1, \ldots, m^r n^r\}$ such that $\gcd(i, m^r n^r)$ is m^r times an r-th power free integer. In particular, for all divisors d of n, we see that $J_r(d)$ is the number of elements $i \in \{1, \ldots, n^r\}$ such that $\gcd(i, n^r)$ is $(\frac{n}{d})^r$ times an r-th power free integer. Hence we have

$$n^r = \sum_{d \mid n} J_r(d) \tag{1}$$

for all positive integers n, as desired.
Let $\iota_r(s)$ denote the Dirichlet series of J_r, i.e. $\iota_r(s) = \sum_{i=1}^{\infty} \frac{J_r(i)}{i^s}$. We show that if $k > r + 1$, then $E_{k,r}(n)$ converges to $\iota_r(k)$ as $n \to \infty$. Let n be any positive integer. Then we have

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$$\left|\sum_{1\le y\le n}\frac{J_r(y)}{y^k}-E_{r,k}(n)\right| = n^{-k}\sum_{1\le y\le n}J_r(y)\Big((\tfrac{n}{y})^k-\lfloor\tfrac{n}{y}\rfloor^k\Big)$$
$$\le n^{-k}\sum_{1\le y\le n}y^r\Big((\tfrac{n}{y})^k-(\tfrac{n}{y}-1)^k\Big).$$

As the function $\mathbb{R}_{>0}\to\mathbb{R}_{>0}$, $x\mapsto x^k$ is convex, we have $(x-1)^k > x^k - kx^{k-1}$ for all real numbers $x>0$. Therefore we have

$$\left|\sum_{1\le y\le n}\frac{J_r(y)}{y^k}-E_{r,k}(n)\right| \le n^{-k}\sum_{1\le y\le n}\Big(y^r k(\tfrac{n}{y})^{k-1}\Big)$$
$$= kn^{-1}\sum_{1\le y\le n}y^{r-k+1}$$
$$\le kn^{-1}\Big(1+\int_1^n y^{r-k+1}\,\mathrm{d}y\Big).$$

In the last step, we use that $k>r+1$, so $r-k+1<0$, and that hence the integrand is decreasing. Moreover, as $r-k+1<0$, we have

$$\int_1^n y^{r-k+1}\,\mathrm{d}y = o(n),$$

so therefore $E_{r,k}(n)$ converges to $\iota_r(k)$ as $n\to\infty$.
Next, we compute $\iota_r(k)$ for $k>r+1$. Note that by (1), we have

$$\zeta(k-r)=\sum_{n=1}^{\infty}\frac{n^r}{n^k}=\Big(\sum_{n=1}^{\infty}\frac{J_r(n)}{n^k}\Big)\Big(\sum_{n=1}^{\infty}\frac{1}{n^k}\Big)=\iota_r(k)\zeta(k),$$

so $\iota_r(k)=\frac{\zeta(k-r)}{\zeta(k)}$.
In particular, the desired variance is, for $k>3$,

$$\iota_2(k)-\iota_1(k)^2=\frac{\zeta(k-2)\zeta(k)-\zeta(k-1)^2}{\zeta(k)^2}.$$

Problem 2014-4/B (folklore)
The evil Eve has locked Alice and Bob in a room without windows. Outside the room, there is a corridor with 64 doors. Eve puts a key behind one of the doors and a crocodile behind each of the others. Then she hangs up a light bulb above each of the doors, and for each light bulb, switches it on or off. Then Eve brings Alice into the corridor, tells her which door hides the key and tells her to choose one of the light bulbs and change the state of that chosen light bulb. After Alice leaves, Eve brings Bob to the corridor, and tells him to open a door of his own choice. Alice and Bob are allowed to discuss a strategy before Alice is shown where the key is, but not after.

a. Give a strategy that guarantees Bob to find the key.
b. For which positive integers n does such a strategy exist if there are n doors?

Solution We received solutions from Alex Heinis, Julian Lyczak and Raymond van Bommel, Thijmen Krebs, Gerard Renardel, Jos van Kan and Kees Vugs. The following solution is based on that of Julian Lyczak and Raymond van Bommel and they win the book token.
We show that there exists a strategy for Alice and Bob if and only if n is a power of 2. This solves both parts a and b.
First suppose that there exists a strategy for Alice and Bob for an n that is not a power of 2. Bob's part of the strategy is a map from the set of 2^n possible configurations of light bulbs to the set of n doors. By the pigeon hole principle, there is a door k that is reached by at most $\lfloor 2^n/n\rfloor$ light bulb configurations (here $\lfloor x\rfloor$ is x rounded down to an integer). As Alice has to choose between n light bulbs, she can reach these configurations from at most $n\lfloor 2^n/n\rfloor$

starting configurations. As n is not a divisor of 2^n, we have $n\lfloor 2^n/n\rfloor < 2^n$, so there is a starting configuration for which the strategy does not work.

Now suppose $n = 2^d$. We give the following strategy. Alice and Bob first agree on a numbering of the doors, each with a label in $\mathbf{F}_2^d$, where $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ (equivalently, each door gets a number from 0 to $n-1$ of which we take the binary representation). We take sums in $\mathbf{F}_2^d$ componentwise modulo 2, that is, the sum is the bit-wise xor.

When Alice enters the corridor, she computes the sum $s \in \mathbf{F}_2^d$ of the doors whose lights are on. Let $k \in \mathbf{F}_2^d$ be the door with the key. Then Alice changes the state of door $s+k$. After that change, the sum of the doors whose lights are on is $s + (s + k) = k$. When Bob arrives, he computes the sum of the doors whose lights are on and thus finds k, hence opens door k.

Problem 2014-4/C (folklore)

Let X_3 be the collection of three-element subsets of $\{1, 2, \dots, 8\}$, and let X_4 be the collection of four-element subsets of $\{1, 2, \dots, 11\}$. Does there exist an injective map $\phi\colon X_3 \to X_4$ with the following properties?

1. For all subsets $V \subseteq X_3$, we have $\#(\bigcap_{v\in V} \phi(v)) \geq \#(\bigcap_{v\in V} v)$.
2. For all $v, v' \in X_3$, if $v \cap v' = \emptyset$, then $\phi(v) \cap \phi(v') = \emptyset$.

Solution We received solutions from Raymond van Bommel and Julian Lyczak, Alex Heinis and Thijmen Krebs. The book token goes to Thijmen Krebs. The following solution is based on his.

The answer to the question is "no". Suppose for a contradiction that there does exist such a map ϕ.

We first show that we may assume that $v \subset \phi(v)$ for all $v \in X_3$. Consider for $i \in \{1, 2, \dots, 8\}$ the set $V_i = \{v \in X_3 : i \in v\}$. By property 1, $\bigcap_{v\in V_i} \phi(v)$ is non-empty. Choose for every $i \in \{1, 2, \dots, 8\}$ an element $s(i)$ of $\bigcap_{v\in V_i} \phi(v)$. We claim that s is an injective map $\{1, 2, \dots, 8\} \to \{1, 2, \dots, 11\}$. Indeed, for any $i \neq i' \in \{1, 2, \dots, 8\}$, we can find disjoint $v, v' \in X_3$ with $i \in v$ and $i' \in v'$. By property 2, $\phi(v)$ and $\phi(v')$ are disjoint, so in particular distinct, from which we deduce that $s(i) \neq s(i')$, which proves our claim. Therefore we can extend s to a permutation σ of $\{1, 2, \dots, 11\}$. So, replacing ϕ by $v \mapsto \sigma^{-1}\big[\phi(v)\big]$ if necessary, we may indeed assume that for all $v \subset X_3$, we have $v \subset \phi(v)$.

Let us do so. Then we see that for all $v \in X_3$, we have $\phi(v) = v \cup \big\{i(v)\big\}$ for some $i(v) \in \{1, 2, \dots, 11\}$. We claim that for all $v \in X_3$, we in fact have $i(v) \in \{9, 10, 11\}$. Indeed, otherwise we can find disjoint $v, v' \in X_3$ such that $i(v) \in v'$, but then we have a contradiction with (2).

Therefore we have a map $i\colon X_3 \to \{9, 10, 11\}$ satisfying the following properties.

1′. For all $v \in X_3$, we have $\phi(v) = v \cup \{i(v)\}$.

2′. For all disjoint $v, v' \in X_3$, we have $i(v) \neq i(v')$.

So now it suffices to show that no maps $i\colon X_3 \to \{9, 10, 11\}$ satisfying property 2′ exist.

We do this by showing there exist four elements of X_3 which map to pairwise distinct elements.

For any $w \subseteq \{1, 2, \dots, 8\}$, define $V_w = \{v \in X_3 : v \subseteq w\}$.

Lemma 1. *Let $w, w' \subseteq \{1, 2, \dots, 8\}$ be two subsets defining a partition of $\{1, 2, \dots, 8\}$. Then i is constant on either V_w or $V_{w'}$.*

Proof. For all $v \in V_w$ and $v \in V_{w'}$, we see that v and v' are disjoint, so by (2'), we have $i(v) \neq i(v')$. Therefore $i[V_w]$ and $i[V_{w'}]$ are disjoint subsets of $\{9, 10, 11\}$, so i is constant on either V_w or $V_{w'}$. □

Choose any collection W of four subsets $w \subseteq \{1, 2, \dots, 8\}$ such that $\#w = 4$ and such that for all distinct $w, w' \in W$, we have $\#(w \cap w') = 2$, so that $\#(w \cup w') = 6$. Such W exist, take for example

$$W = \Big\{\{1,2,3,4\}, \{1,2,5,6\}, \{1,3,5,7\}, \{1,4,5,8\}\Big\}.$$

Note that W retains this property if we replace any element w by its complement. Therefore by Lemma 1, we may assume, by replacing some of the elements $w \in W$ by their complement if necessary, that V_w is constant for all $w \in W$. So let us do so, and write $i(w)$ for the unique element of $i[V_w]$, for all $w \in V_w$.

Note that by construction, for all distinct $w, w' \in W$, we can find disjoint $v, v' \in X_3$ such that $v \in V_w$ and $v' \in V_{w'}$. Therefore $i(w) \neq i(w')$ for all distinct $w, w' \in W$, in other words, i defines an injection from a four-element set W to a three-element set $\{9, 10, 11\}$, which is impossible. Therefore no ϕ satisfying properties 1 and 2 can exist.