Problemen

Problem Section

Edition 2014-3 We received solutions from Pieter de Groen (Brussels), Thijmen Krebs (Nootdorp), Tejaswi Navilarekallu and Hendrik Reuvers (Maastricht).

All three problems of 2014/3 asked for a construction with origami in a limited number of moves. More precisely, given a collection of points and lines (or *folds*) in the plane, a *move* (cf. the Huzita–Justin–Hatori axioms) consists of adding to the collection one of the following:

- a fold aligning two distinct points;
- a fold aligning two distinct lines;
- if it exists, a fold having two properties of the following types (except for type 3, one may have two distinct alignments of the same type):
  1. the fold aligns a point with a line;
  2. the fold passes through a point;
  3. the fold is perpendicular to a line;
- a sufficiently general fold having at most one property of types 1, 2 and 3.

Problem 2014/3-A. Given three points A, B and C, and a line l passing through C, construct in at most six moves a point D on the line l such that $|CD| = |AB|$.

Solution We received solutions from Pieter de Groen, Thijmen Krebs, Tejaswi Navilarekallu and Hendrik Reuvers. The book token goes to Thijmen Krebs. The following is based on his solution. We give a solution in five moves.

We first assume that AC is not perpendicular to l, and that A does not lie on l.

- Make the fold l_1 aligning A with C.

Let $E = l \cap l_1$. (This uses the assumption that AC is not perpendicular to l.)

- Make the fold l_2 through A and E.
- Make a fold l_3 through A aligning B with l_2.
- Make the fold l_4 through B perpendicular to l_3.

Let $B' = l_2 \cap l_4$. (If $l_2 = l_4$, take $B' = B$ instead.) As l_3 is an angular bisector of $\angle BAB'$, and $l_4 = BB'$ is perpendicular to l_3, it follows that $\triangle ABB'$ is isosceles with apex A. Therefore $|AB'| = |AB|$.

- Make the fold l_5 through B' perpendicular to l_1.

Let $D = l \cap l_5$. (This uses the assumption that A does not lie on l.) As l_1 and l_5 both are perpendicular to AC, by the previous argument, it follows that $|CD| = |AB'| = |AB|$, as desired.

If in the above case, A lies on l, but B does not, then we can simply switch the roles of A and B in the above.

Now assume that either AC is perpendicular to l, or that both A, B lie on l.

- Make the fold l_1 aligning A with C.
- Make a fold l_2 aligning both A and B with l_1.

Let $E = l_1 \cap l_2$. (Note that l_1 is parallel to l_2 if and only if AB is parallel to l, but in that case we could have constructed D in one move in the first place.)

- Make the fold l_3 through B perpendicular to l_2.

Let $F = l_1 \cap l_3$.

- Make the fold l_4 through E aligning C with l_1, so that l_4 is the reflection of l_2 in l_1.
- Make the fold l_5 through F perpendicular to l_4.

Let $D = l \cap l_5$. Moreover, let A' be the auxiliary point that is the reflection of A in l_2. Then, arguing in a similar way as in the previous case, we see that $|A'F| = |AB|$ in both cases. If A, B lie on l, then we also have $|CD| = |A'F| = |AB|$ by the same argument. If AC is perpendicular to l, then l and l_1 are parallel, and so are $A'C$ and l_5; so $CDFA'$ is a parallelogram, and $|CD| = |A'F| = |AB|$.

Problem 2014/3-B. Construct a golden rectangle (including its sides) in at most eight moves.

Solution We received solutions from Pieter de Groen, Thijmen Krebs, Tejaswi Navilarekallu and Hendrik Reuvers. The book token goes to Tejaswi Navilarekallu. The following is based on his solution, which has similar ideas to those of Pieter de Groen and Thijmen Krebs.

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Oplossingen | Solutions

- Make a fold l_1.
- Make a fold l_2 perpendicular to l_1.
- Make a fold l_3 (distinct from l_2) perpendicular to l_1.
- Make the fold l_4 aligning l_1 and l_2.

Let $A = l_1 \cap l_2$, $B = l_1 \cap l_3$, and let $X = l_3 \cap l_4$. Then $\triangle ABX$ is an isosceles triangle with apex B, so $|AB| = |BX|$.

- Make the fold l_5 aligning B and X.

Let $Y = l_3 \cap l_5$. Then $|BY| = \frac{1}{2}|BX| = \frac{1}{2}|AB|$, so $|AY| = \frac{1}{2}\sqrt{5}$.

- Make the fold l_6 through Y aligning A with l_3 such that A and B lie on the same side of l_6.
- Make the fold l_7 through A perpendicular to l_6.

Let $C = l_3 \cap l_7$. As l_6 is an angular bisector of $\angle AYC$, and l_7 is perpendicular to l_6, the triangle $\triangle ACY$ is isosceles with apex Y. Therefore $|CY| = |AY| = \frac{1}{2}\sqrt{5}$, and $|BC| = |BY| + |CY| = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, i.e. A, B, C form three vertices of a golden rectangle.

- Make the fold l_8 through C perpendicular to l_3.

Now $ABCD$ is a golden rectangle (with sides l_1, l_2, l_3, l_8).

Problem 2014/3-C. Given two points A and B, construct in at most four moves the point C on the segment AB such that $|AC| = \frac{1}{3}|AB|$.

Solution We received solutions from Pieter de Groen, Thijmen Krebs, Tejaswi Navilarekallu and Hendrik Reuvers. The book token goes to Hendrik Reuvers. The following solution is based on that of Thijmen Krebs.

- Make the fold l_1 aligning A with B.
- Make a fold l_2 aligning A with l.

Let $P = l_1 \cap l_2$. Let A' be the reflection of A in l. (Note that A' is not a point that we have constructed; we only use it as an auxiliary point for the following arguments.) Then $|A'B| = |AB|$, and as l is the perpendicular bisector of AB, it follows that $AA'B$ is equilateral, and that P is its centroid. Therefore, if B' is the midpoint of AA', we have $|PB'| = \frac{1}{3}|BB'|$.

- Make the fold l_3 through P perpendicular to l_2.
- Make the fold l_4 through A and B.

Let $C = l_3 \cap l_4$. As l_3 is parallel to AA', it follows that $|AC| = \frac{1}{3}|AB|$, as desired.

Rectifications

i. In the solution of Problem 2014/3-C that appeared in March this year (see above), the symbol l occurs twice without a subscript. The first one should be l_1, and the second one should be l_2.

ii. We omitted by mistake some references, submitted to us by Rob van der Waall.
   - For the problems of the September issue of 2014, [1] and [2] are two books related to the subject of the problems.
   - Problem 2014/2-C appears as exercise 3.28 in [3]. Its solution also follows from Theorem VI.II of [4].

We apologise for the mistakes made.

References

1. T. Sundara Row, *Geometric Exercises in Paper Folding*, Dover Publications, New York, 1966 (original year of publication: 1893).
2. W. W. Beman and D. E. Smith, *New Plane and Solid Geometry*, Ginn and Co., Boston, 1899.
3. John D. Dixon, *Problems in Group Theory*, Dover Publ., 1967.
4. W. Burnside, *Theory of Groups of Finite Order*, second edition, Dover Publ., 1955.