Problem
Problem Section

Edition 2014-2 We received solutions from Rik Bos (Bunschoten), Pieter de Groen (Brussels), Alex Heinis (Amsterdam), Nicky Hekster (Amstelveen), Carlo Pagano (Leiden) and Rob van der Waall (Huizen).

Problem 2014-2/A (proposed by Wouter Zomervrucht)
Let n be a positive integer. Let M be an $n \times n$-matrix with entries in $\{1, 2, \ldots, n\}$. Let r be the complex eigenvalue with the largest absolute value. Show that $n \le |r| \le n^2$.

Solution We received correct solutions from Rik Bos, Pieter de Groen, Alex Heinis, Carlo Pagano and Rob van der Waall. The book token is awarded to Rob van der Waall.
Let $\|\cdot\|$ be the Frobenius matrix norm (i.e. for $M = (m_{ij})_{i,j=1}^n$, we have $\|M\|^2 = \sum_{i=1}^n \sum_{j=1}^n |m_{ij}|^2$). Gelfand's formula states

$$|r| = r = \lim_{k\to\infty} \|M^k\|^{1/k}.$$

Let N be another matrix as in the problem, with largest eigenvalue s. Suppose $M \le N$, i.e. inequality on all entries, then also $M^k \le N^k$, so $\|M^k\| \le \|N^k\|$, and $r \le s$. Let M_0 be the $n \times n$-matrix with all entries equal to 1. It has eigenvalues 0 and n. Note that nM_0 has eigenvalues 0 and n^2. We have $M_0 \le M \le nM_0$, hence $n \le r \le n^2$ as desired.

Problem 2014-2/B (proposed by Hans Zwart)
Let X be a unital $\mathbb{R}$-algebra with multiplicative unit 1, and let $\|\cdot\|$ be a *submultiplicative norm* on X, i.e. a map $\|\cdot\|\colon X \to \mathbb{R}$ satisfying the following properties:
- $\|1\| = 1$;
- if $x \in X$ satisfies $\|x\| = 0$, then $x = 0$;
- for all $a \in \mathbb{R}, x \in X$, we have $\|ax\| = |a|\,\|x\|$;
- for all $x, y \in X$, we have $\|x + y\| \le \|x\| + \|y\|$ and $\|xy\| \le \|x\|\,\|y\|$.

Let $C\colon \mathbb{R} \to X$ be a map such that $C(0) = 1$ and such that for all $s, t \in \mathbb{R}$, we have

$$2C(s)C(t) = C(s+t) + C(s-t).$$

Suppose that

$$\sup_{s\in\mathbb{R}} \|C(s) - 1\| < \tfrac{3}{2}.$$

Show that $C = 1$.

Solution We received no correct solutions. The following solution is based on that of the proposer, Hans Zwart.
Let $t \in \mathbb{R}$. Then we have

$$C(2t) - 1 = 2C(t)^2 - C(0) - 1 = 2C(t)^2 - 2 = 2\Big(C(t) - 1\Big)^2 + 4\Big(C(t) - 1\Big).$$

Hence for all $t \in \mathbb{R}$, we have $4\Big(C(t)-1\Big) = \Big(C(2t)-1\Big) - 2\Big(C(t)-1\Big)^2$, so if $\rho = \sup_{s\in\mathbb{R}} \|C(s)-1\|$, then for all $t \in \mathbb{R}$, we have

$$\begin{aligned} 4\|C(t) - 1\| &= \left\|\Big(C(2t) - 1\Big) - 2\Big(C(t) - 1\Big)^2\right\| \\ &\le \|C(2t) - 1\| + 2\|C(t) - 1\|^2 \\ &\le \rho + 2\rho^2, \end{aligned}$$

hence $4\rho \le \rho + 2\rho^2$, or equivalently, $\rho(2\rho - 3) \ge 0$. As we assumed that $\rho < \frac{3}{2}$, it follows that $\rho = 0$, hence $C(t) = 1$ for all $t \in \mathbb{R}$, as desired.

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Oplossingen

Solutions

Problem 2014/2-C (proposed by Hendrik Lenstra)
Let G be a finite group. Let n be the number of automorphisms σ of G such that for all $x \in G$, the element $\sigma(x)$ is conjugate to x. Show that every prime divisor of n divides the order of G.

Solution We received solutions from Rik Bos, Alex Heinis, Nicky Hekster, Carlo Pagano and Rob van der Waall. All their solutions were along the same lines, which we reproduce here. The book token goes to Alex Heinis.

Let A be the set of $\sigma \in$ Aut(G) such that for all $x \in G$, the element $\sigma(x)$ is conjugate to x, so $n = \#A$. Let p be a prime divisor of n. Note that A is a subgroup of Aut(G), hence by Cauchy's theorem has an element σ of order p. This element σ acts on each conjugacy class of G, and the orbits have length 1 or p.

Let $H = \{x \in G : \sigma(x) = x\}$ be the union of the orbits of length 1, which is a proper subgroup of G.

Lemma. *Let $H \subset G$ be a subgroup with $H \neq G$. Then the union of the conjugates of H is not G.*

Proof. The group H has at most $[G : H]$ conjugates, because gHg^{-1} depends only on $gH \in G/H$. These conjugates all have $\#H$ elements and have the element 1 in common, hence the union of the conjugates has fewer than $[G : H] \cdot \#H = \#G$ elements. □

By the lemma, we see that there is an element of $x \in G$ that is not conjugate to any element of H. Its conjugacy class C is a union of orbits for $\langle\sigma\rangle$, which all have length p as C is disjoint with H, so $\#C$ is divisible by p.

By the orbit-stabilizer theorem, the order of G is divisible by the length of the conjugation orbit C of x. □