Problemen

Problem Section

Edition 2014-1 We received solutions from Rik Bos (Bunschoten), Charles Delorme, Pieter de Groen (Brussels, Belgium), Alex Heinis (Amsterdam), Nicky Hekster (Amstelveen), Alexander van Hoorn (Abcoude), Huub van Kempen (The Hague), Thijmen Krebs (Nootdorp), Hendrik Reuvers (Maastricht), Traian Viteam (Cape Town, South Africa), Robert van der Waall (Huizen) and Sander Zwegers (Cologne, Germany).

Problem 2014-1/A (proposed by Hendrik Lenstra)
Let G be a group, and let $a, b \in G$ be two elements satisfying $\{gag^{-1} : g \in G\} = \{a, b\}$. Prove that for all $c \in G$ one has $abc = cba$.

Solution We received solutions from Rik Bos, Pieter de Groen, Alex Heinis, Nicky Hekster, Alexander van Hoorn, Thijmen Krebs, Traian Viteam, Robert van der Waall and Sander Zwegers. All their solutions were along the same lines. The book token goes to Nicky Hekster.
The case $a = b$ is easy, so we assume $a \neq b$. We have $bab^{-1} \in \{a, b\}$ and $bab^{-1} = b$ would contradict $a \neq b$, hence we have $ba = ab$.
Note that $\{a, b\}$ is an orbit under the conjugation action, hence conjugation by c acts either by swapping a and b or by fixing both a and b.
In the latter case, the element c commutes with both a and b, which also commute with each other, hence $abc = cba$.
In the former case, we have $abc = a(cac^{-1})c = aca = c(c^{-1}ac)a = cba$.

Problem 2014-1/B (due to Albrecht Pfister [1])
Let K be a field, and consider for all positive integers n the subset S_n of $x \in K^*$ that can be written as the sum of n squares in K. Show that the subgroup of K^* generated by S_n is equal to $S_{t(n)}$. Here, for a positive integer n, we denote by $t(n)$ the smallest power of two that is greater than or equal to n.

Solution We received solutions by Rik Bos, Alex Heinis, Thijmen Krebs and Robert van der Waall. They all used or referred to Pfister's lemma. The book token goes to Thijmen Krebs.
For any $n \geq 1$ the set S_n contains 1, and if $x_1^2 + \cdots + x_n^2$ is non-zero, then

$$(x_1^2 + \cdots + x_n^2)^{-1} = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}\right)^2 + \cdots + \left(\frac{x_n}{x_1^2 + \cdots + x_n^2}\right)^2.$$

Pfister's lemma. *Let $n = 2^m$. For all $x \in S_n$ there exists an $n \times n$-matrix X such that $XX^\top = X^\top X = xI_n$.*
Proof. By induction on m. It is obvious for $m = 1$, so suppose the lemma is true for some $m \geq 1$. Any $x \in S_{2n}$ either lies in S_n, or is of the form $y + z$ with $y, z \in S_n$. In the first case, there is nothing to prove. In the second case, let Y, Z be matrices with $YY^\top = Y^\top Y = yI_n$ and $ZZ^\top = Z^\top Z = zI_n$. They exist by assumption. Let X be the block matrix

$$X = \begin{matrix} Y & Z \\ -(Y^{-1}ZY)^\top & Y^\top \end{matrix}.$$

Then $XX^\top = X^\top X = xI_n$ as desired. □

Let n be a power of 2. Take $x, y \in S_n$, and let X, Y be corresponding matrices. Then $Z = XY^\top$ satisfies $ZZ^\top = xyI_n$. If $z_1, \ldots, z_n$ is the first row of Z, this implies $xy = z_1^2 + \cdots + z_n^2$, hence S_n is closed under multiplication. It follows that S_n is a group.
It now suffices to show that for any $n \geq 1$ we have $S_{t(n)} \subseteq \langle S_n \rangle$. Write $t(n) = 2r$. Any $x \in S_{t(n)}$ is either in $S_r \subseteq \langle S_n \rangle$, or can be written as $y + z$ with $y, z \in S_r$. Then $x = y(1 + z/y)$ is the product of $y \in S_r$ and $1 + z/y \in S_{r+1}$, using that z/y lies in S_r since S_r is a group. It follows that x is in $\langle S_n \rangle$.

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Oplossingen

Solutions

Problem 2014-1/C (folklore)

Given five pairwise distinct points A, B, C, D, E in the plane, no three of which are collinear, and given a line l in the plane not passing through any of the five points. Assume that l intersects the conic section c passing through A, B, C, D, E. Construct the intersection points of l and c.

Solution We received solutions from Charles Delorme (17 moves), Huub van Kempen (19 moves), and Hendrik Reuvers (20 moves). The book token goes to Charles Delorme. This solution below is based on that of Charles Delorme, Hendrik Reuvers and [2].

Rename the 5 points to A_1, A_2, A_3, X_1, X_2. The construction is then as follows.

– For all $i \in \{1,2,3\}$ and $j \in \{1,2\}$, draw the line A_iX_j. (6 moves)

Let B_{ij} be the intersection of A_iX_j with l, for $i \in \{1,2,3\}$ and $j \in \{1,2\}$. Note that the desired intersection points c with l then are the fixed points of the projectivity on l sending B_{i1} to B_{i2} for $i \in \{1,2,3\}$. The idea is now to project these points to a circle, and then use Steiner's double element construction (see e.g. [2]).

– Draw the circle Γ with centre X_2 passing through X_1. (1 move)

– For all $i \in \{1,2,3\}$, draw the line $B_{i2}X_1$. (3 moves)

Let C_{ij} be the intersection of $B_{ij}X_1$ with Γ, for $i \in \{1,2,3\}$ and $j \in \{1,2\}$. Note here that, for $i \in \{1,2,3\}$, $B_{i1}X_1 = A_iX_1$, a line that we have already drawn. We will now perform Steiner's double element construction.

– Draw the lines $C_{11}C_{32}$, $C_{12}C_{31}$, $C_{21}C_{32}$, $C_{22}C_{31}$. (4 moves)

Let $P_1 = C_{21}C_{32} \cap C_{22}C_{31}$ and $P_2 = C_{11}C_{32} \cap C_{12}C_{31}$.

– Draw P_1P_2. (1 move)

Let Q_1 and Q_2 denote the two intersections of P_1P_2 with Γ.

– Draw Q_1X_1 and Q_2X_1. (2 moves)

Let R_i ($i \in \{1,2\}$) denote the intersection of Q_iX_1 with l. We have now used 17 moves. Moreover, by Steiner's double element construction, Q_1 and Q_2 were the fixed points of the projectivity such that $C_{i1} \mapsto C_{i2}$ for all $i \in \{1,2,3\}$, hence R_1 and R_2 are the intersection points of c with l, as desired.

References

1. Albrecht Pfister, Zur Darstellung von -1 als Summe von Quadraten in einem Korper, *J. London Math. Soc.* 40 (1965), 159–165.

2. H. Dörrie, *100 Great Problems of Elementary Mathematics, their History and Solution* (translation of Thriumph der Mathematik, 1932), reworked in 2010 by M. Woltermann.