Problemen

Problem Section

Edition 2013-4 We received solutions from Alexandros Efthymiadis (St. Andrews), Alex Heinis (Amsterdam), Richard Kraaij (Delft), Thijmen Krebs (Nootdorp), Matthé van der Lee (Amsterdam), Tejaswi Navilarekallu, Rohith Varma (Chennai, India), Traian Viteam (Cape Town, South Africa) and Hans Zwart (Enschede).

Problem 2013-4/A (folklore, communicated by Jaap Top)
Does there exist an integer $n > 1$ such that the *set* of leading digits of $2^n, 3^n, \ldots, 9^n$ is equal to $\{2, 3, \ldots, 9\}$?

Solution The answer to the question is 'no'. Let $\mathrm{ld}(k)$ denote the leading digit of k. The more general question which patterns occur in the table

$$\left(\mathrm{ld}(2^n), \mathrm{ld}(3^n), \ldots, \mathrm{ld}(9^n)\right)_n$$

is known as Gelfand's question. See, e.g., mathworld.wolfram.com/GelfandsQuestion.html.
We received correct solutions from Alexandros Efthymiadis, Alex Heinis and Thijmen Krebs. The book token goes to Thijmen Krebs, and the following solution is based on his.
noindent Suppose $n > 1$ as in the problem exists and let $\sigma_d = \mathrm{ld}(d^n)$, so we have a permutation $d \mapsto \sigma_d$ of $\{2, 3, \ldots, 9\}$. Then for every d, there exists some $m_d \in \mathbb{Z}$ with

$$\sigma_d 10^{m_d} \leq d^n < (\sigma_d + 1)10^{m_d}. \tag{1}$$

In particular, we have

$$\sigma_2\sigma_5 10^{m_2+m_5} \leq 10^n < (\sigma_2 + 1)(\sigma_5 + 1)10^{m_2+m_5}, \tag{2}$$

so $n = m_2 + m_5 + 1$ and $\sigma_2\sigma_5 \in \{6, 8, 10\}$.
In case $\sigma_2\sigma_5 = 10$, we have equality in the left hand side of (2), hence $5^n = \sigma_5 10^{m_5}$, but the left hand side has no factors 2, so $m_5 = 0$ and $n = 1$, contradiction. This leaves the three cases $\sigma_2 = 2, 3, 4$.
If $\sigma_2 = 4$, then $\sigma_5 = 2$, but also $\sigma_4 = 2$ because of

$$\sigma_2^2 10^{2m_2} \leq 4^n < (\sigma_2 + 1)^2 10^{2m_2}. \tag{3}$$

A contradiction since $\sigma_2 \neq \sigma_4$.
If $\sigma_2 = 3$, then $\sigma_5 = 2$ and $\sigma_4 = 9$ because of (3). But then $\sigma_8 \in \{2, 3\}$ because of

$$\sigma_2\sigma_4 10^{m_2+m_4} \leq 8^n < (\sigma_2 + 1)(\sigma_4 + 1)10^{m_2+m_4}. \tag{4}$$

This is a contradiction as $\sigma_2, \sigma_5, \sigma_8 \in \{2, 3\}$ are distinct.
Finally, if $\sigma_2 = 2$, then $\sigma_5 \in \{3, 4\}$ and $\sigma_4 \leq 4$ by 4 (otherwise $\sigma_8 = 2 = \sigma_2$). Similarly, the inequality

$$\sigma_2\sigma_3 10^{m_2+m_3} \leq 6^n < (\sigma_2 + 1)(\sigma_3 + 1)10^{m_2+m_3} \tag{5}$$

implies $\sigma_3 \leq 4$. Now $\sigma_2, \sigma_3, \sigma_4, \sigma_5 \in \{2, 3, 4\}$ is another contradiction.
This proves that n does not exist and solves the problem.
Wolfram MathWorld (mathworld.wolfram.com/GelfandsQuestion.html) calls the question whether there is an n with $\mathrm{ld}(d^n) = d$ for all $d \in \{2, 3, \ldots, 9\}$ an open question. The solution above implies that the answer to this question is 'no' as well.

Problem 2013-4/B (proposed by Bart de Smit and Hendrik Lenstra)
Rings are unital, and morphisms of rings send 1 to 1. Let A and B be commutative rings.

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Suppose that there exists a ring C such that there are injective morphisms $A \to C$ and $B \to C$ of rings. Show that there exists a *commutative* such ring.

Solution We received solutions from Matthé van der Lee and Rohith Varma. The book token is awarded to Matthé van der Lee.
Let $f\colon A \to C$ and $g\colon B \to C$ be the given injections. The map

$$A \times B \to C, \quad (a,b) \mapsto f(a)g(b)$$

is $\mathbb{Z}$-bilinear, so induces a ring homomorphism $h\colon A \otimes_{\mathbb{Z}} B \to C$. Let $i\colon A \to A \otimes_{\mathbb{Z}} B$ and $j\colon B \to A \otimes_{\mathbb{Z}} B$ be the canonical maps. The compositions $f = h \circ i$ and $g = h \circ j$ are injective by assumption, hence i and j are injective. The ring $A \otimes_{\mathbb{Z}} B$ is commutative, so we are done.

Problem 2013-4/C (proposed by Jinbi Jin)
Let $C(\mathbb{R},\mathbb{R})$ denote the set of continuous maps from $\mathbb{R}$ to itself. A (not necessarily continuous) map $f\colon C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ is called *good* if it satisfies, for all $s,t \in C(\mathbb{R},\mathbb{R})$, the identity

$$f(s \circ t) = f(s)f(t),$$

where the product on the right hand side is the point-wise multiplication of maps.

- Find a non-constant good map $f\colon C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$.
- Show that $f(\exp) = 0$ for all non-constant good maps $f\colon C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$. (Here, exp is given by $x \mapsto e^x$.)

Solution We received solutions from Alex Heinis, Richard Kraaij, Thijmen Krebs, Tejaswi Navilarekallu, Traian Viteam and Hans Zwart. The following solution (of the second part) is based on that of Tejaswi Navilarekallu, to whom we also award the book token for this problem.

Giving a non-constant good map
Let $f\colon C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ be given by

$$f(s) = \begin{cases} 1 & \text{s is bijective,} \\ 0 & \text{otherwise.} \end{cases}$$

Showing that f is indeed a good map amounts to proving the following lemma.

Lemma 1. *Let $s,t\colon \mathbb{R} \to \mathbb{R}$ be continuous maps such that $s \circ t$ is bijective. Then both s and t are bijective.*

Proof. First note that t and $s \circ t$ are injective, so by the intermediate value theorem, they are either strictly increasing, or strictly decreasing. Assume without loss of generality that t and $s \circ t$ is strictly increasing.
Suppose for a contradiction that the image of t is bounded from above, and let M be its supremum in $\mathbb{R}$. Let $x \in \mathbb{R}$ be arbitrary, and consider the interval $I = [x, \infty)$. As t is increasing, it follows that the image of I under t lies in $[t(x), M]$. As $s \circ t$ is increasing and bijective, it follows that $s \circ t(I) = [s \circ t(x), \infty)$. But on the other hand, as $[t(x), M]$ is closed and bounded, $s \circ t(I)$ must be a closed and bounded interval, and this is a contradiction. Hence the image of t is not bounded from above, and similarly, we can show that it is not bounded from below either. As t is a strictly increasing continuous map, it follows that t is a bijection, and hence that s is a bijection as well, as desired. □

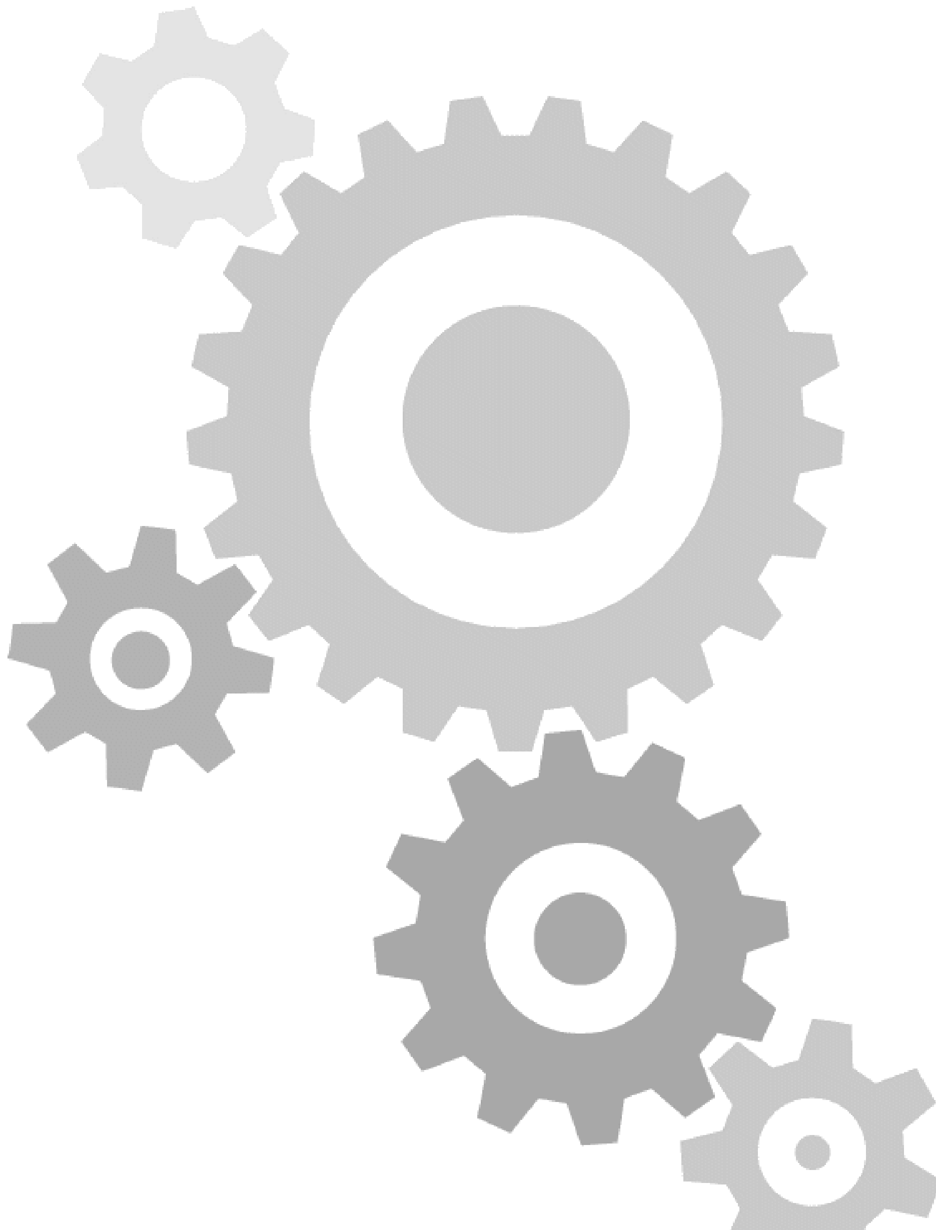
Showing that $f(\exp) = 0$ for f non-constant and good
Let f be a non-constant good map. In what follows below we denote by 0, 1 and -1 the constant maps taking the corresponding values.
Let

$$i(x) = x, \ j(x) = -x, \ u(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For any $s \in C(\mathbb{R},\mathbb{R})$ such that $s \circ s = s$ we have $f(s) = f(s \circ s) = f(s)f(s)$. Since $f(s)$ is a continuous function, it follows that $f(s) = 1$ or $f(s) = 0$.
Note that $i \circ i = i$, so $f(i) = 1$ or $f(i) = 0$. If $f(i) = 0$, then $f(s) = f(s)f(i) = 0$ for all $s \in C(\mathbb{R},\mathbb{R})$ and thus f is constant, contrary to our assumption. Therefore $f(i) = 1$. Also note that $0 \circ 0 = 0$, so $f(0) = 1$ or $f(0) = 0$. If $f(0) = 1$, then $f(s) = f(0 \circ s) = 1$ for all $s \in C(\mathbb{R},\mathbb{R})$, so f is again constant, contrary to our assumption. Hence $f(0) = 0$. Finally, note that $f(i) = f(j \circ j) = f(j)^2$ and hence $f(j) = 1$ or $f(j) = -1$.
Now

$$\begin{aligned} f(u)f(j) &= f(u \circ j) \\ &= f(u \circ u \circ j) = f(u)f(u \circ j) = f(u \circ j)f(u) = f(u \circ j \circ u) \\ &= f(0) = 0. \end{aligned}$$

Since $f(j) = \pm 1$, it follows that $f(u) = 0$. Therefore, as $u \circ \exp = \exp$, we have $f(\exp) = f(u \circ \exp) = f(u)f(\exp) = 0$.



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Erratum to solution 2013-4/B. The map $h\colon A \otimes_{\mathbb{Z}} B \to C$ is not necessarily a *ring* homomorphism; it is just a $\mathbb{Z}$-module homomorphism. Nevertheless, the natural maps $i\colon A \to A \otimes_{\mathbb{Z}} B$ and $j\colon B \to A \otimes_{\mathbb{Z}} B$ are ring homomorphism, so the proof is unaffected.

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