Problemen

Problem Section

Edition 2013-3 We received solutions from Pieter de Groen (Brussels), Alex Heinis (Amsterdam), Tejaswi Navilarekallu and Sander Zwegers (Keulen).

Problem 2013-3/A (proposed by Hendrik Lenstra)
Let $a, b \in \mathbb{C}$. Show that if there exists an irreducible polynomial $f \in \mathbb{Q}[X]$ such that $f(a) = f(a+b) = f(a+2b) = 0$, then $b = 0$.

Solution We received solutions from Alex Heinis, Tejaswi Navilarekallu and Sander Zwegers. The following solution is based on that of Alex Heinis, who will receive the book token.
Let S be the (finite) set of complex zeros of f and let H be the convex hull of S, which is a polygon or a line segment. Let $v \in S$ be a vertex of H.
Since f is irreducible, the Galois group G of its splitting field acts transitively on S, hence there is an element $\sigma \in G$ with $\sigma(a+b) = v$.
Now $\sigma(a)$ and $\sigma(a+2b)$ are also in S, and $v = \sigma(a+b)$ lies on the line segment connecting them. This contradicts the fact that v is a vertex, unless $b = 0$.

Problem 2013-3/B (a result due to B. Konstant and N. Wallach [1])
Let n be a positive integer, and let e_{ij} be an integer for all $1 \le j \le i \le n$. Show that there exists an $n \times n$–matrix with entries in $\mathbb{Z}$ such that the eigenvalues of the top left $i \times i$–minor are $e_{i1}, \ldots, e_{ii}$ (with multiplicity).

Solution We received correct solutions from Pieter de Groen, Alex Heinis and Sander Zwegers. All gave a solution similar to the one we present below. The book token goes to Sander Zwegers.
For any square matrix A we denote the top left $i \times i$–minor by $m_i(A)$. We also use the notation $\chi_A(\lambda) := \det(\lambda I - A)$ for the characteristic polynomial of A. Let S_n be the set of all $n \times n$–matrices $A = (a_{ij})$ over the polynomial ring $\mathbb{Z}[\lambda]$ such that $a_{i+1,i} = 1$ for all $1 \le i < n$ and such that $a_{i,j} = 0$ if $i \ge j+2$, i.e., A has ones on the line just below its main diagonal and has zeroes below this line of ones. (We allow λ in the matrices because we will work with characteristic polynomials.) It suffices to prove the following claim:

Claim. *Let n be a positive integer and let e_{ij} be an integer for all $1 \le j \le i \le n$. Then there exists a matrix $A \in S_n$ with entries in $\mathbb{Z}$ such that $\chi(m_i(A)) = (\lambda - e_{i1}) \cdots (\lambda - e_{ii})$ for all $1 \le i \le n$.*

We prove this claim by induction on n. The base case $n = 1$ is trivial: the matrix (e_{11}) will do. Now suppose that the claim is true for some $n = N-1 \ge 1$ and let us prove it for $n = N$. So let e_{ij} be an integer for all $1 \le j \le i \le N$ and let $A \in S_{N-1}$ be the matrix we get from the claim for $n = N-1$ and the integers e_{ij} with $1 \le j \le i \le N-1$. Since $1, \chi(m_1(A)), \ldots, \chi(m_{N-1}(A)), \lambda\chi(m_{N-1}(A))$ are monic polynomials over $\mathbb{Z}$ of degree $0, 1, \ldots, N-1, N$ (respectively), they form a $\mathbb{Z}$–basis for the abelian group of polynomials of degree $\le N$ with integer coefficients. So there exist integers $x_1, \ldots, x_{N+1}$ such that the degree–N polynomial $(\lambda - e_{N1}) \cdots (\lambda - e_{NN})$ is equal to the linear combination

$$-x_1 + (-1)^{N+1} x_{N+1} \lambda \chi(m_{N-1}(A)) + \sum_{i=2}^{N} (-1)^i x_i \chi(m_{i-1}(A)).$$

Since this polynomial is monic of degree N, we know that $x_{N+1} = (-1)^{N+1}$. We claim that we can take the following matrix to prove the claim for $n = N$:

$$B = \begin{pmatrix} & & & x_1 \\ & A & & x_2 \\ & & & \vdots \\ 0 & \cdots & 1 & x_N \end{pmatrix}.$$

First of all, $A \in S_{N-1}$ implies that $B \in S_N$. Moreover, for $i < N$ we have $\chi(m_i(B)) = \chi(m_i(A)) = (\lambda - e_{i1}) \cdots (\lambda - e_{ii})$ by the construction of B and the choice of A. So we are left to compute $\chi(m_N(B)) = \chi(B)$.

Redactie:
Johan Bosman
Gabriele Dalla Torre
Christophe Debry
Jinbi Jin
Marco Streng
Wouter Zomervrucht

Problemenrubriek NAW
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

Oplossingen | Solutions

Lemma. *For every* $M = (M_{ij}) \in S_n$ *(*$n \geq 2$*) we have*

$$\det(M) = (-1)^n \left(-M_{1,n} + \sum_{i=2}^{n} (-1)^i M_{i,n} \det(m_{i-1}(M)) \right).$$

Proof. Expand the determinant with respect to the last row to find that

$$\det(M) = M_{n,n} \det(m_{n-1}(M)) - \det \begin{pmatrix} & & M_{1,n} \\ & m_{n-2}(M) & M_{2,n} \\ & & \vdots \\ 0 & \cdots \quad 1 & M_{n-1,n} \end{pmatrix}.$$

Now use induction and the fact that $m_i(m_j(M)) = m_i(M)$ for all $i \leq j \leq n$. □

The lemma enables us to compute the characteristic polynomial of B:

$$\begin{aligned} \chi(B) = (-1)^N \det(B - \lambda I) &= -x_1 + \sum_{i=2}^{N} (-1)^i (B - \lambda I)_{i,N} \det(m_{i-1}(B - \lambda I)) \\ &= -x_1 + x_2 \chi(m_1(A)) + \cdots + (-1)^{N-1} x_{N-1} \chi(m_{N-2}(A)) \\ &\quad + (-1)^N (x_N - \lambda) \chi(m_{N-1}(A)). \end{aligned}$$

By definition of $x_1, \ldots, x_N$, this polynomial is equal to $(\lambda - e_{N1}) \cdots (\lambda - e_{NN})$ and this concludes the proof of the claim for $n = N$.

Problem 2013-3/C (proposed by Bart de Smit and Hendrik Lenstra)
Let A be a finite commutative unital ring. Does there exist a pair (B, f) with B a finite commutative unital ring in which every ideal is principal, and f an injective ring homomorphism $A \to B$?

Solution We received no correct solutions. The answer to the question is "no". Let A be the finite commutative unital ring $\mathbb{F}_2[x, y]/(x^2, y^2)$. Let B be a finite commutative unital ring in which every ideal is principal, and let $f : A \to B$ be a ring homomorphism. It now suffices to show that f cannot be injective.
First note that the ring homomorphism f forces $2 = 0$ in B. By assumption there exists an element $z \in B$ such that we have the identity $\big(f(x), f(y)\big) = (z)$ as ideals of B. Hence there exist $b, c \in B$ such that $f(x) = bz$ and $f(y) = cz$, so $f(xy) = bcz^2$. Writing $z = df(x) + ef(y)$ with $d, e \in B$, we get $z^2 = d^2 f(x^2) + 2def(xy) + e^2 f(y^2) = 0$. We deduce that $f(xy) = 0$, hence f is not injective, as desired.

Reference

1. B. Kostant and N. Wallach, Gelfand–Zeitlin theory from the perspective of classical mechanics, Studies in Lie theory, *Progress in Mathematics* 243, 319–364, http://arxiv.org/abs/math/0408342v2.