Problemen

| Problem Section

Edition 2013-2 We received solutions from Leon van den Broek (Nijmegen), Alex Heinis (Amsterdam), Jos van Kan (Delft), Thijmen Krebs (Nootdorp), Javier Sánchez-Reyes (Castilla-La Mancha, Spain) and Ángel Plaza (Las Palmas de Gran Canaria, Spain), and Robert van der Waall (Huizen).

Problem 2013-2A (based on a problem proposed by Gerard Renardel de Lavalette)
We have two hourglasses, A for a seconds and B for b seconds, where a and b are relatively prime integers and $0 < a < b$. Let t_0 be an integer with $t_0 \geq b + (\frac{1}{2}a - 1)^2$. Show that A and B can be used to identify the time $t = t_0$ if the upper bulbs are empty at $t = 0$.

Remark. *The original problem received from the proposer was to prove a slightly stronger result. Let m be the remainder of b upon division by a. The original problem was to prove that for any integer $t_0 > b + m(a - m) - a$, the time $t = t_0$ can be identified using A and B.*

Solution We received only one correct solution, from Thijmen Krebs, who will receive the book token. The following solution is based on that solution.
Let m be the remainder of b upon division by a. For any integer T that is a multiple of a or b, we can use the following strategy:

- while $t < T$, turn each hourglass whenever it is empty;
- while $t \geq T$, turn *both* hourglasses whenever at least one is empty.

If we apply this strategy to $T = b$, then we turn both hourglasses at the times $t = b + km$ for $k = 0, 1, 2, \ldots$.
If we apply this strategy to $T = a(1 + \frac{b-m}{a}) = b + (a - m)$, then we turn both hourglasses at the times $t = b + k(a - m)$ for $k = 1, 2, 3, \ldots$.
In particular, all elements of the following set are measurable times:

$$S = \{b + km : 0 \leq k < a - m\} \cup \{b + k(a - m) : 0 < k \leq m\}.$$

As a and b are coprime, so are m and a, hence S contains an element of each residue class modulo a. Moreover, the maximal element of S is $b + m(a - m)$.
Before starting the strategy above, we can measure any non-negative integer multiple of a seconds using A, while letting B stay empty. In particular, we can measure any time $t_0 \geq b + m(a - m) - a + 1$.
Finally, note $m(a - m) \leq (\frac{1}{2}a)^2$, so $b + m(a - m) - a + 1 \leq b + (\frac{1}{2}a)^2 - a + 1 = b + (\frac{1}{2}a - 1)^2$ and we can measure any time $t_0 \geq b + (\frac{1}{2}a - 1)^2$.

Problem Problem 2013-2B (folklore, communicated by Jeanine Daems)
In a two-player game, players take turns drawing a number of coins from a pile that starts with n coins. The first player takes at least one coin from the pile, but not all. In the subsequent turns, each player takes at least one coin, and at most twice the number of coins taken in the previous turn. The player who takes the last coin wins. For which numbers n can the first player win?

Solution We received correct solutions from Alex Heinis and Thijmen Krebs. The book token is awarded to Alex Heinis. The game is known as *Fibonacci Nim*, and the first player can win for those integers $n > 1$ that are not a Fibonacci number.
Let $(F_k)_{k\geq1}$ be the Fibonacci sequence: $F_1 = 1$, $F_2 = 2$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 1$. The proof uses *Zeckendorf's theorem*: every positive integer can uniquely be written as the sum of non-consecutive Fibonacci numbers. Let z be the function on the positive integers that assign to m the smallest Fibonacci number occurring in the Zeckendorf decomposition of m. E.g., we can write $20 = 13 + 5 + 2 = F_6 + F_4 + F_2$ and $z(20) = F_2 = 2$.
We define a position in this game to be a pair (m, d) where m is number of coins left on the pile and d the maximal number of coins that may be taken (by the player who is to move). The initial position is $(n, n - 1)$ and the final positions are those of the form $(0, d)$. Call a position (m, d) 'good' if it is non-final and $d \geq z(m)$; call it 'bad' otherwise.

Lemma. *Let (m, d) be a good position. There exists a move to a bad position.*

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Proof. Write $m = F_{k_1} + \cdots + F_{k_r}$ for the Zeckendorf decomposition, with $k_i \geq k_{i+1} + 2$. By assumption, d is at least F_{k_r}. Our move is to take exactly F_{k_r} coins. The new position is $(m - F_{k_r}, 2F_{k_r})$. This is a bad position: in the case $r = 1$ it is even final, and otherwise it follows from $2F_{k_r} < F_{k_{r-1}}$. □

Lemma. *Let (m, d) be a non-final bad position. All moves lead to a good position.*

Proof. Write $F_k = z(m)$. By assumption we have $d < F_k$. Suppose we take x coins, for some $x \in \{1, \ldots, d\}$. Let $t \geq 0$ be the even number such that $F_{k-t-2} \leq x < F_{k-t}$. Then

$$F_k - F_{k-t} < F_k - x \leq F_k - F_{k-t-2}$$

hence

$$F_{k-1} + F_{k-3} + \cdots + F_{k-t+1} < F_k - x \leq F_{k-1} + F_{k-3} + \cdots + F_{k-t-1},$$

so $z(F_k - x) \leq F_{k-t-1}$, which is smaller than $2F_{k-t-2} \leq 2x$. Note further that $z(F_k - x) = z(m - x)$. Hence $(m - x, 2x)$ is a good position. □

Together the lemmas show that the good positions are exactly the winning ones. The initial position $(n, n-1)$ is good if and only if $z(n) \leq n - 1$, i.e., if and only if n is not Fibonacci.

Problem Problem 2013-2C (proposed by Bas Edixhoven and Maarten Derickx)
Let $ABCD$ be a convex quadrilateral inside a plane U in $\mathbb{R}^3$. Suppose that $ABCD$ is not a parallelogram. Show that there exist a plane V in $\mathbb{R}^3$ and a point $P \in \mathbb{R}^3 - (U \cup V)$ such that if a light source is placed in P, then the shadow of $ABCD$ on V is a square.

Solution We received solutions from Leon van den Broek, Alex Heinis, Jos van Kan, Javier Sánchez-Reyes and Ángel Plaza, and Robert van der Waall. The book token goes to Jos van Kan.
The main idea of the solution is to pick the plane V and the point P in such a way that the projection on V of the intersection of the lines AB and CD, and that of BC and AD is 'at infinity'. Some extra conditions on P related to the diagonals and consecutive edges will then ensure that the projection of $ABCD$ on V is a square.
If the lines AB and CD intersect, denote their intersection by X_1. Similarly, if BC and AD intersect, denote their intersection by X_2.
We consider three cases, the first of which is the following.

Case 1. The lines AB and CD intersect, and so do BC and AD. Moreover, both of AC and BD intersect the line X_1X_2.

First note that X_1X_2 does not intersect the quadrilateral $ABCD$, as $ABCD$ is convex. Let Y_1 be the intersection of AC and X_1X_2, and likewise, let Y_2 be the intersection of BD and X_1X_2.
Let W be a plane that has as intersection the line X_1X_2 with U. In particular, $U \neq W$. Let Γ_1, Γ_2 be the circles in W with the segments $\overline{X_1X_2}$, $\overline{Y_1Y_2}$ as diameter, respectively. Let P be an intersection of Γ_1 and Γ_2, and let V be any plane parallel to W such that the quadrilateral $ABCD$ lies between V and W. This intersection exists as one of Y_1, Y_2 lies between X_1 and X_2, and the other does not.
Then note that P does not lie in U, as the points X_1, X_2, Y_1, Y_2 are pairwise distinct, and that P does not lie in V, as P lies in W, which is parallel to V. Hence $P \in \mathbb{R}^3 - (U \cup V)$.
Now let A_0, B_0, C_0, D_0 be the respective intersections of AP, BP, CP, DP with V. They exist, as the given lines intersect in P with W, which is parallel to V. By construction of V, and as X_1X_2 does not intersect the quadrilateral $ABCD$, it now suffices to show that $A_0B_0C_0D_0$ is a square in V.
Let l be a line in U, not equal to X_1X_2. Write $\mathcal{P}(l)$ for the unique plane through l and P, write $\mathcal{I}(l)$ for the intersection line of W with $\mathcal{P}(l)$, and write $\mathcal{I}_0(l)$ for the intersection line of V with $\mathcal{P}(l)$. (So for example, $\mathcal{L}_0(AB) = A_0B_0$.) As V and W are parallel, it follows that for all lines l, m in U, the angle between $\mathcal{I}(l)$ and $\mathcal{I}(m)$ is equal to the one between $\mathcal{I}_0(l)$ and $\mathcal{I}_0(m)$. Note that a square is a quadrilateral such that

– every two successive edges are perpendicular, and
– the diagonals are perpendicular.

Therefore, to show that $A_0B_0C_0D_0$ is a square, it suffices to show that the $\mathcal{I}(e)$ (with e an edge or a diagonal of the quadrilateral $ABCD$) satisfy the above properties.
Now we simply note that

$$\mathcal{I}(AB) = \mathcal{I}(CD) = PX_1, \qquad \mathcal{I}(BC) = \mathcal{I}(AD) = PX_2,$$

and that

$$\mathcal{I}(AC) = PY_1, \qquad \mathcal{I}(BD) = PY_2,$$

so by construction of P, the quadrilateral $A_0B_0C_0D_0$ is a square, as desired.
For the remaining cases, we will only state them, and the corresponding construction of P, as the proof (and the construction of V) is done in the same way.

Case 2. The lines AB and CD intersect, and so do BC and AD. Moreover, at most one of AC and BD intersects the line X_1X_2.

Note here that at least one of AC and BD intersects the line X_1X_2, as AC and BD intersect, so exactly one of them intersects X_1X_2. We assume without loss of generality that AC and X_1X_2 intersect, and let Y be their intersection. Note that Y lies between X_1 and X_2, as the line AC intersects the segment $\overline{BD}$, which is parallel to X_1X_2. Let W be any plane that has as intersection the line X_1X_2 with U, and let Γ be the circle in W with diameter $\overline{X_1X_2}$. Then we take P to be an intersection of Γ with the line through Y perpendicular to X_1X_2.

Case 3. Exactly one of the pairs of lines (AB, CD) and (BC, AD) intersect.

We assume without loss of generality that AB and CD intersect, and let X denote this intersection. Let l be the line through X parallel to BC (hence also to AD). Then the lines AC and BD both intersect l, as they intersect BC. Let Y_1 and Y_2 be their respective intersections. Then X lies between Y_1 and Y_2, as for S the intersection of AC and BD, the line XS intersects the segments $\overline{BC}$ and $\overline{AD}$, which are parallel to l. Let W be any plane that has l as intersection with U, and let Γ be the circle with diameter $\overline{Y_1Y_2}$. Then we take P to be an intersection of Γ with the line through X perpendicular to l.

References. This problem turned out to be rather well-known, as we received a lot of references. Thanks to Leon van den Broek, Javier Sánchez-Reyes and Ángel Plaza, and Robert van der Waall for these. The references given were, respectively,

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