# Problemen | Problem Section

**Edition 2013-1** We received solutions from Wouter Cames van Batenburg (Leiden), Hao Chen (Seattle), Charles Delorme (Paris), Florian Eisele (Brussels), Alex Heinis (Hoofddorp), Pieter de Groen (Brussels), Thijmen Krebs (Nootdorp), Guido Senden (Groningen), Sep Thijssen (Nijmegen) and Traian Viteam (Punta Arenas).

**Problem 2013-1/A** Consider a regular $n$-gon $P_1P_2\dots P_n$, and draw $n-3$ diagonals such that there are no intersection points in the interior. The polygon is now divided into $n-2$ triangles. Let $t_i$ be the number of such triangles that have a vertex at $P_i$. Show that

$$t_1 - \cfrac{1}{t_2 - \cfrac{1}{\dots - \cfrac{1}{t_{n-1}}}} = 0.$$

**Solution** We received solutions from Charles Delorme, Alex Heinis, Pieter de Groen, Guido Senden, Sep Thijssen, Traian Viteam and Thijmen Krebs. The book token goes to Sep Thijssen. All solutions followed the same general strategy. The following solution is most similar to that of Charles Delorme, Pieter de Groen and Sep Thijssen.

We use $[t_1,\dots,t_{n-1}]$ to denote the continued fraction from the problem statement, and we will prove the identity by induction for all convex (not necessarily regular) $n$-gons with $n \geq 3$. Note that the base case $n=3$ is trivial as $1-\frac{1}{1}=0$.

Now assume $n \geq 4$ and assume that the identity holds for $n-1$. The induction step works by finding a triangle with two exterior edges, that is, a triangle $P_{k-1}P_kP_{k+1}$ with $t_k = 1$, and by removing that triangle from the $n$-gon, which yields an $(n-1)$-gon.

**Lemma 1.** *If $n \geq 4$, then there exists an integer $k$ with $2 \leq k \leq n-1$ and $t_k = 1$.*

*Proof.* We prove first that there are at least two integers $k$ with $1 \leq k \leq n$ with $t_k = 1$. Suppose that there is at most one such integer, that is, suppose that there is at most one triangle with two exterior edges. The total number of exterior edges $n$ then is at most $2 + (\#\text{triangles} - 1) = n-1$, which is a contradiction.

Now note that no two such integers $k$ can correspond to adjacent vertices, so that at least one of them satisfies $2 \leq k \leq n-1$. □

Now let $k$ be as in the lemma and assume for now $k \neq n-1$. Consider the $(n-1)$-gon with the vertex $P_k$ and the edges $P_{k-1}P_k$ and $P_kP_{k+1}$ removed. For that $(n-1)$-gon, the triangle count $t_k$ is omitted, and the adjacent numbers $t_{k-1}$ and $t_{k+1}$ are lowered by 1. In other words, the induction hypothesis yields

$$[t_1,\dots,t_{k-2},t_{k-1}-1,t_{k+1}-1,t_{k+2},\dots,t_{n-1}] = 0.$$

In particular, the proof of the case $k \neq n-1$ is finished once we prove the following key identity.

**Lemma 2.** *We have*

$$\begin{aligned}&[t_1,\dots,t_{k-2},t_{k-1}\phantom{{}-1},1,t_{k+1}\phantom{{}-1},t_{k+2},\dots,t_{n-1}]\\ =&[t_1,\dots,t_{k-2},t_{k-1}-1,\phantom{1,}t_{k+1}-1,t_{k+2},\dots,t_{n-1}].\end{aligned}$$

*Proof.* From the definition, we directly get

$$[a_1,\dots,a_m,[b_1,\dots,b_n]] = [a_1,\dots,a_m,b_1,\dots,b_n]. \qquad (1)$$

Moreover, it is a simple matter of writing things out to see

$$[a,1,b] = [a-1, b-1]. \tag{2}$$

Indeed, we have

$$a - \frac{1}{1-\frac{1}{b}} = a - \frac{b}{b-1} = a - 1 - \frac{1}{b-1}.$$

The result follows by taking $a = t_{k-1}$ and $b = [t_{k+1}, t_{k+2}, \ldots, t_{n-1}]$ in (2) and applying (1) repeatedly. □

This leaves only the case $k = n-1$, for which we use the identity

$$[t_1, \ldots, t_{k-2}, t_{k-1}, 1] = [t_1, \ldots, t_{k-2}, t_{k-1} - 1],$$

which is obvious as the tail reads $t_{k-1} - \frac{1}{1} = t_{k-1} - 1$.

**Remark.** One might worry about division by zero. However, if one takes $1/0 = \infty$ and $1/\infty = 0$, then all the identities in the proof above continue to make sense even with divisions by zero. Alternatively, it is possible to prove that division by zero does not occur. This can be done by showing in the induction $[t_i, \ldots, t_{n-1}] \geq 0$ for $i = 1, \ldots, n-1$ with equality if and only if $i = 1$, which requires a few extra case distinctions in the induction step.

**Remark.** If one does the induction with $k = 1$ or $k = n$, then the induction step becomes more complicated. Fortunately we did not need to do this thanks to Lemma 1. For example, with $k = 1$, one needs to prove the implication

$$[t_2 - 1, t_3, \ldots, t_{n-1}] = 0 \quad \Longrightarrow \quad [1, t_2, t_3, \ldots, t_{n-1}] = 0.$$

Let $b = [t_2, t_3, \ldots, t_{n-1}]$, then this reads $b - 1 = 0 \Rightarrow 1 - 1/b = 0$, which is true, but cannot be proved by an identity as in the cases $2 \leq k \leq n-1$, because generally $b - 1 \neq 1 - 1/b$.

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**Problem 2013-1/B** You are allowed to transform positive integers $n$ in the following way. Write $n$ in base 2. Write plus signs between the bits at will (at most one per position), and then perform the indicated additions of binary numbers. For example, $123_{10} = 1111011$ can get + signs after the second, third and fifth bits to become $11 + 1 + 10 + 11 = 9_{10}$; or it can get + signs between all the bits to become $1 + 1 + 1 + 1 + 0 + 1 + 1 = 6_{10}$; and so on.
Prove that it is possible to reduce arbitrary positive integers to $1$ in a bounded number of steps. That is, there is a constant $C$ such that for any $n$ there is a sequence of at most $C$ transformations that starts with $n$ and ends at $1$.

**Solution** This problem appeared originally in the Fall 2011 edition of the MSRI Emissary. We received solutions from Wouter Cames van Batenburg, Hao Chen, Pieter de Groen, Thijmen Krebs and Guido Senden. The book token goes to Guido Senden. The following solution contains ideas from various solutions.
For a positive integer $n$, let $I(n)$ denote the number of $1$'s in its binary expansion. For two positive integers $a, b$, we denote by $(ab)_2$ the positive integer obtained by concatenating the binary expansions of $a$ and $b$. We remark that if $a, b$ can be transformed into $x, y$, respectively, then $(ab)_2$ can be transformed into $x + y$, by placing a + between $a$ and $b$.

**Lemma 3.** *Every positive integer $n$ can be transformed into every integer $x$ with $I(n) \leq x \leq \frac{3}{2}I(n)$.*
*Proof.* One transforms $n$ into $I(n)$ by placing + at every position. Omitting a + after a $1$ that has a + before it raises the sum by $1$, as $2 = (1+0)+1$ and $3 = (1+1)+1$. This can be done at least $\left\lfloor \frac{1}{2} I(n) \right\rfloor$ times, by omitting the plus after every other $1$. □

**Corollary 4.** *Any positive integer $n$ such that $\frac{2}{3}2^{\lceil \log_2 I(n) \rceil} \leq I(n) \leq 2^{\lceil \log_2 I(n) \rceil}$ can be transformed into $2^{\lceil \log_2 I(n) \rceil}$.*

The following proposition, in combination with Corollary 4 now shows that we only need to consider the cases where $I(n) \in \{4, 5, 8, 9, 10\}$.

**Proposition 5.** *Suppose that all positive integers $n$ with $8 \leq I(n) \leq 16$ can be transformed into 16. Let $k \geq 4$, and let $n$ be a positive integer with $2^{k-1} \leq I(n) \leq 2^k$. Then $n$ can be transformed into $2^k$.*

*Proof.* We proceed by induction. For $k = 4$, this is exactly our assumption. Suppose that the proposition is true for $k = i$, and let $n$ be a positive integer with $2^i \leq I(n) \leq 2^{i+1}$. Write $n = (ab)_2$ with $I(a) = \left\lfloor \frac{1}{2} I(n) \right\rfloor$ and $I(b) = \left\lceil \frac{1}{2} I(n) \right\rceil$. Then $2^{i-1} \leq I(a) \leq I(b) \leq 2^i$, so $a, b$ can both be transformed into $2^i$, hence $n$ can be transformed into $2^{i+1}$, as desired. □

We will now first treat the easiest of the remaining cases, i.e. $I(n) \in \{4, 8, 10\}$.

**Lemma 6.** *Any positive integer $n$ such that $I(n) = 4$ can be transformed into 8.*

*Proof.* Write $n = (ab)_2$ with $a$ consisting of the first three bits of $n$. If $a$ starts with $(11)_2$, then $a = 8 - I(b)$, and $b$ can be transformed into $I(b)$ by Lemma 3, so $n$ can be transformed into 8. If $a$ starts with $(10)_2$, then $a = 7 - I(b)$, and $b$ can be transformed into $I(b) + 1$ by Lemma 3, as $I(b) \geq 2$. We deduce that $n$ can be transformed into 8. □

**Corollary 7.** *Any positive integer $n$ such that $I(n) \in \{8, 10\}$ can be transformed into 16.*

*Proof.* Write $n = (ab)_2$, with $I(a) = 4$. Then both $a$ and $b$ can be transformed into 8 by Lemma 6 and Corollary 4, so $n$ can be transformed into 16. □

Finally, we do the case $I(n) \in \{5, 9\}$, which requires considering more cases.

**Lemma 8.** *Any positive integer $n$ such that $I(n) = 5$ can be transformed into a power of 2. If $n$ does not start with $(11)_2$, then $n$ can be transformed into 8.*

*Proof.* First suppose that $n$ starts with $(11)_2$. Then write $n = (ab)_2$, where $a$ consists of the first four bits of $n$. If $a$ starts with $(111)_2$, then $a = 16 - I(b)$, and $b$ can be transformed into $I(b)$ by Lemma 3, so $n$ can be transformed into 16. If $a$ starts with $(110)_2$, then $a = 15 - I(b)$, and $b$ can be transformed into $I(b) + 1$ by Lemma 3, as $I(b) \geq 2$. Hence $n$ can be transformed into 16. Otherwise $n$ starts with $(10)_2$. In this case, write $n = (ab)_2$, where $a = (10)_2$ consists of the first two bits of $n$. Now $I(b) = 4$, which can be transformed into 6 by Lemma 3, so $n$ can be transformed into 8. □

**Lemma 9.** *Any positive integer $n$ such that $I(n) = 9$ can be transformed into 16.*

*Proof.* First suppose that $n$ starts with $(11)_2$. Write $n = (ab)_2$ with $a$ consisting of the first three bits of $n$. Then $a = 4 + I(a) = 13 - I(b)$. As $I(b) \geq 6$, we have $3 \leq \frac{1}{2} I(b)$. Hence $I(b) \leq I(b) + 3 \leq \frac{3}{2} I(b)$, so $b$ can be transformed into $I(b) + 3$. We deduce that $n$ can be transformed into 16.

Otherwise $n$ starts with $(10)_2$. Write $n = (ab)_2$ in this case, with $a$ containing the first 5 ones of $n$, and $b$ containing the last 4 ones. Then by Lemmas 6 and 8, $a$ and $b$ can both be transformed into 8, so $n$ can be transformed into 16. □

**Remark.** As Pieter de Groen and Guido Senden remarked, one can prove very quickly that any positive integer can be reduced to 1 in at most 3 steps, once one has Corollary 4. For this, note that the remaining cases are those for which $2^{\lceil \log_2 I(n) \rceil - 1} < I(n) < \frac{2}{3} 2^{\lceil \log_2 I(n) \rceil}$. The smallest such case is $I(n) = 5$. Then one can transform $n$ into $\frac{3}{2} \cdot 2^{\lceil \log_2 I(n) \rceil - 1}$ instead by Lemma 3, as $\frac{3}{2} > \frac{4}{3}$, so

$$I(n) < \frac{2}{3} \cdot 2^{\lceil \log_2 I(n) \rceil} = \frac{4}{3} \cdot 2^{\lceil \log_2 I(n) \rceil - 1} < \frac{3}{2} \cdot 2^{\lceil \log_2 I(n) \rceil - 1} < \frac{3}{2} I(n).$$

The number $\frac{3}{2} \cdot 2^{\lceil \log_2 I(n) \rceil - 1}$ can then be transformed into 2 as $I(n) \geq 5$, which can then be transformed into 1, showing that any positive integer can be reduced to 1 in at most three steps.

# Oplossingen | Solutions

**Problem 2013-1/C** Let $R$ be a commutative ring with $1$. Consider the set

$$S = \left\{(i, j) \in R^2 : i^2 = i, j^2 = j, ij = 0\right\}.$$

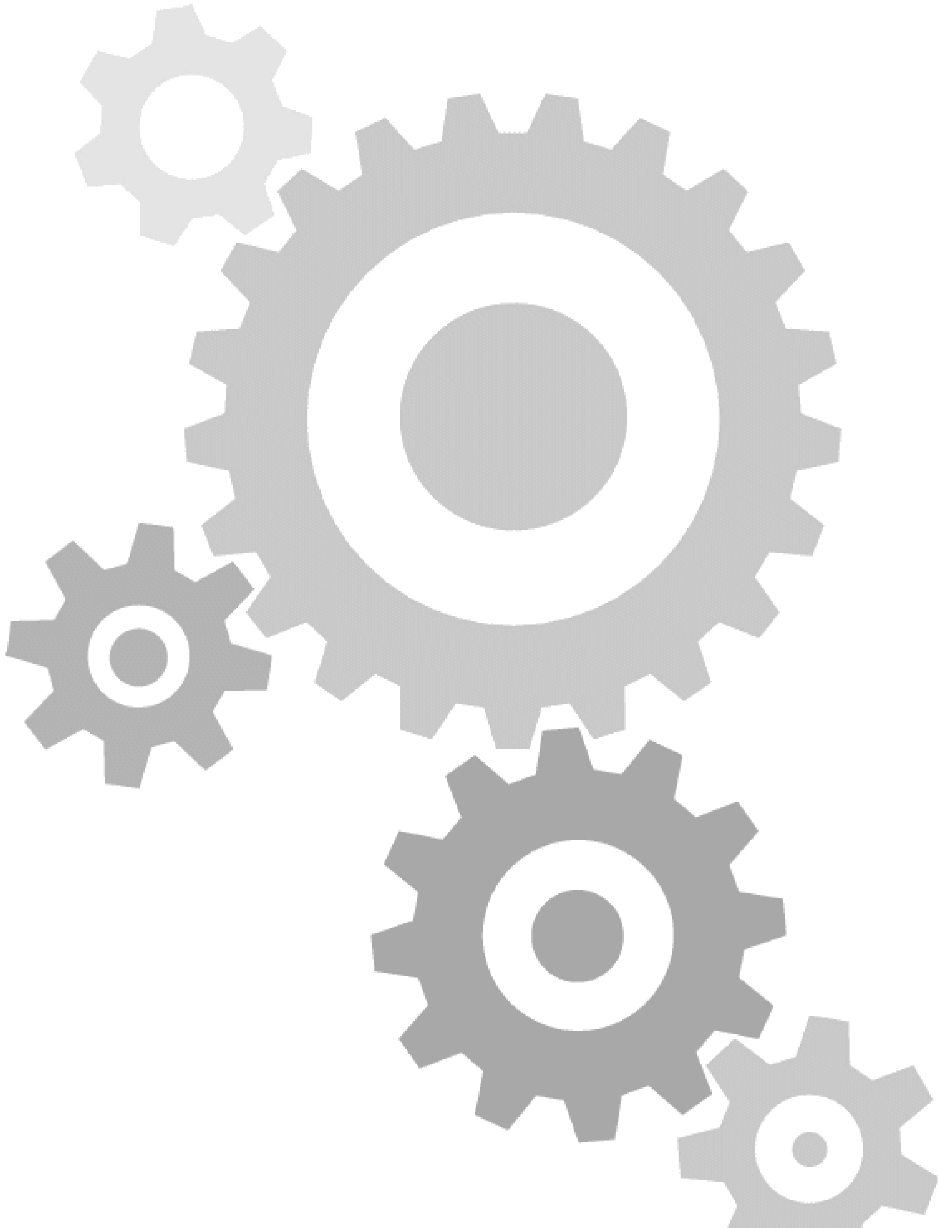
Show that the cardinality of $S$ is a power of $3$ if $S$ is finite.

**Solution** We received solutions from Florian Eisele, Thijmen Krebs and Guido Senden. The book token goes to Florian Eisele. The following solution is partly based on that of Florian Eisele and Thijmen Krebs.

Let us write $S(R)$ for the set $S$ corresponding to a (commutative unital) ring $R$. First note that for all idempotents $i \in R$ (i.e. elements $i \in R$ with $i^2 = i$) we have $(i, 0) \in S(R)$. Hence the set of idempotents $I(R)$ of $R$ is finite if $S$ is finite, so we can proceed by induction on the number of idempotents $n(R)$ of $R$. Note that $0, 1 \in R$ both are idempotents for all rings $R$, so $n(R) \geq 1$, and if $n(R) = 1$, then $R$ is the zero ring, for which $S(R) = \{(0, 0)\}$, hence $\#S(R) = 3^0$. If $n(R) = 2$, then $I(R) = \{0, 1\}$, therefore $S(R) = \{(0, 0), (1, 0), (0, 1)\}$, and $\#S(R) = 3^1$.

Now let $k > 2$, and suppose that for all rings $R$ with $n(R) < k$, the set $S(R)$ has cardinality a power of $3$. Let $R$ be a ring with $n(R) = k$. As $n(R) > 2$, there exists an $i \in I(R)$ with $i \notin \{0, 1\}$. Let $j = 1 - i$, then $j \in I(R)$, and $ij = 0$. Then the map $\varphi\colon R \to R/iR \times R/jR, x \mapsto (x + iR, x + jR)$ is an isomorphism of rings by the Chinese Remainder Theorem. Note that $\varphi$ induces a bijection $I(R) \to I(R/iR) \times I(R/jR)$, therefore also a bijection $S(R) \to S(R/iR) \times S(R/jR)$.

As $j$ is non-zero, it induces a non-zero idempotent in $(R/iR) \times (R/jR)$. It follows that $j + iR \neq 0$, so $n(R/iR) > 1$ and similarly (using $i + jR$) also $n(R/jR) > 1$, hence $n(R/iR), n(R/jR) < n(R) = k$. It follows that $\#S(R/iR), \#S(R/jR)$ are powers of $3$, hence so is $\#S(R) = \#S(R/iR)\#S(R/jR)$.



# Problemen

Problem Section

**On Problem 2013-1/A.** We would like to thank Jan Stevens for pointing out that Problem 2013-1/A in fact originated from his work (see [1]), and that there is a connection with Coxeter's *frieze patterns* (see [2, 3]).

## References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*Redactie:*
*Johan Bosman*
*Gabriele Dalla Torre*
*Christophe Debry*
*Jinbi Jin*
*Marco Streng*
*Wouter Zomervrucht*

*Problemenrubriek NAW*
*Mathematisch Instituut*
*Universiteit Leiden*
*Postbus 9512*
*2300 RA Leiden*
*problems@nieuwarchief.nl*
*www.nieuwarchief.nl/problems*