# Problemen

| Problem Section

**Edition 2012-2** We received solutions from Wouter Cames van Batenburg (Leiden), Cor Hurkens (Eindhoven), Thijmen Krebs (Nootdorp), José H. Nieto (Maracaibo) and Hans Zantema (Eindhoven).

**Problem 2012-2/A** Let $P$ and $Q$ be distinct points in the plane. Let $n \geq 2$. Assume $n$ distinct lines through $P$ but not through $Q$ are given, as well as $n$ distinct lines through $Q$ but not through $P$. Let $T$ be a collection of $2n$ intersection points of these lines. Suppose that the (unoriented) angle between the lines $RP$ and $RQ$ is the same for all $R$ in $T$, and not a multiple of $\frac{1}{4}\pi$. Show that $T$ can be partitioned into subsets of at least three elements each, such that every subset consists of the vertices of a regular polygon.

**Rectification.** The common angle in this problem should not be a multiple of $\pi/4$. (Thanks to Thijmen Krebs for pointing this out.)

**Solution** We received a correct solution from Thijmen Krebs.
All angles are oriented angles modulo $\pi$, unless stated otherwise. Let $\alpha$ be the unoriented angle modulo $\pi$ of the common angle of the $\angle PRQ$, where $R \in T$.

**Observation 1.** *Every line through $P$ (resp. $Q$) contains exactly two points of $T$.*

*Proof.* Let $L$ be a line through $P$. As $Q$ is not on this line, there is a unique isosceles triangle with base inside $L$, top $Q$, and base angles $\alpha$. Hence there are at most two points of $T$ on any given line through $P$. But since we have $n$ lines going through $P$, and $\#T = 2n$, it must follow that every line must contain exactly two points of $T$. The same argument holds for $Q$. □

**Observation 2.** *The set $T$ is a subset of the union of two distinct circles intersecting at $P$ and $Q$.*
*Proof.* Note that by the inscribed angle theorem, the subset $T_+$ of $T$ consisting of the points $R \in T$ such that $\angle PRQ = \alpha$ lies on a circle $\Gamma_+$ containing $P$ and $Q$, and that the subset $T_-$ of $T$ consisting of the points $R \in T$ such that $\angle PRQ = -\alpha$ also lies on a circle $\Gamma_-$ containing $P$ and $Q$. Moreover, these circles are distinct since $\alpha \neq \frac{1}{2}\pi$ by assumption. □

We now define two maps $f_P, f_Q \colon T_+ \to T_-$ as follows. Let $R \in T_+$. Then $f_P(R)$ (resp. $f_Q(R)$) is the unique intersection point of the line $RP$ (resp. $RQ$) with $\Gamma_-$ not equal to $P$ (resp. $Q$). This map is well-defined, as for $R \in T_+$, we have $\angle P f_P(R) Q = \angle P f_Q(R) Q = -\alpha$, hence $f_P(R), f_Q(R) \in T_-$ by Observation 1.

**Observation 3.** *The maps $f_P$ and $f_Q$ are bijections. In particular, $\#T_+ = \#T_- = n$.*

*Proof.* We simply note that the inverse is given by sending $R \in T_-$ to the unique intersection point of the line $RP$ (resp. $RQ$) with $\Gamma_+$ not equal to $P$ (resp. $Q$). □

**Observation 4.** *The maps $f_P^{-1} f_Q$ and $f_Q f_P^{-1}$ are rotations by $4\alpha$ (as an oriented angle modulo $2\pi$) on $T_+$ and $T_-$, respectively (with centres those of $\Gamma_+$ and $\Gamma_-$, respectively).*

*Proof.* Let $R \in T_+$. Then $\angle PRQ = \angle Q f_Q(R) P = \alpha$, it follows that $\angle RP f_P^{-1} f_Q(R) = \angle RP f_Q(R) = 2\alpha$. Hence if $C_+$ is the centre of $\Gamma_+$, then $\angle RC_+ f_P^{-1} f_Q(R) = 4\alpha$, as an oriented angle modulo $2\pi$. The same argument works for $f_Q f_P^{-1}$. □

Now we note that the orbits of $T_+$ (resp. $T_-$) under the action of $f_P^{-1} f_Q$ (resp. $f_Q f_P^{-1}$) all have the same length by the above, which hence divides $n$, so it follows that $f_P^{-1} f_Q$ and $f_Q f_P^{-1}$ have order dividing $n$. Hence $4n\alpha = 0$ modulo $2\pi$, so $\alpha = 0$ modulo $\pi/2n$. As we assumed that $\alpha$ was not a multiple of $\frac{1}{4}\pi$, it follows that orbits of length at most 2 cannot occur. Orbits of higher length are sets whose vertices form a regular polygon with at least three vertices, so we are done.

**Problem 2012-2/B** Show that there exist an $n \geq 1$, a polynomial $P \in \mathbb{Z}[X, Y_1, \ldots, Y_n]$ and an infinite set $S$ of positive integers such that the set

*Redactie:*
*Johan Bosman*
*Gabriele Dalla Torre*
*Jinbi Jin*
*Ronald van Luijk*
*Lenny Taelman*
*Wouter Zomervrucht*
*Problemenrubriek NAW*
*Mathematisch Instituut*
*Universiteit Leiden*
*Postbus 9512*
*2300 RA Leiden*
*problems@nieuwarchief.nl*
*www.nieuwarchief.nl/problems*

$$\left\{(y_1,\ldots,y_n)\in\mathbb{Z}^n : P(k,y_1,\ldots,y_n)=0\right\}$$

is empty for all $k<0$ and has precisely $k$ elements for all $k\in S$.

**Solution** We received a correct solution from Thijmen Krebs.
An example can be deduced from Jacobi's four-square theorem. It states that for each positive integer $p$, the number of solutions $(y_1,y_2,y_3,y_4)\in\mathbb{Z}^4$ to

$$y_1^2+y_2^2+y_3^2+y_4^2 = p$$

is $r_4(p)=8\sum_{d\in D} d$, where $D$ is the set of divisors of $p$ that are not multiples of $4$. In particular, if $p$ is prime we have $r_4(p)=8(p+1)$.
Set $n=4$ and let $P\in\mathbb{Z}[X,Y_1,Y_2,Y_3,Y_4]$ be the polynomial

$$P = 8(Y_1^2+Y_2^2+Y_3^2+Y_4^2+1)-X.$$

Define $S=\{8(p+1) : p \text{ prime}\}$. The equation $P(k,y_1,y_2,y_3,y_4)=0$ has no solutions for $k<0$. For $k=8(p+1)\in S$ the equation reduces to $y_1^2+y_2^2+y_3^2+y_4^2=p$, which has $r_4(p)=k$ solutions.

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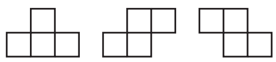
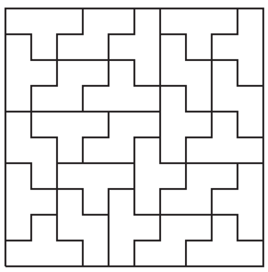
**Problem 2012-2/C** Is it possible to tile a 30 by 30 square grid using the following blocks?



**Solution** We received correct solutions from Wouter Cames van Batenburg, Cor Hurkens, Thijmen Krebs, José H. Nieto and Hans Zantema. The book token goes to José H. Nieto.
There exists a tiling as desired. In fact, we can already tile a $10\times 10$ grid.



Note that we do not even need both types of Z-tiles.
More generally, an $n\times m$ grid can be tiled with the given pieces if and only if $n$ and $m$ are at least $4$, $nm$ is divisible by $4$, and $(n,m)$ is not $(6,6)$, $(6,10)$ or $(10,6)$.