Problemen

| Problem Section

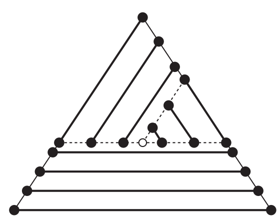
Edition 2012-1 We received solutions from: Charles Delorme (Paris), Pieter de Groen (Brussels), Alex Heinis (Hoofddorp), Ruud Jeurissen (Nijmegen), Thijmen Krebs (Nootdorp), Paolo Perfetti (Rome), Merlijn Staps (Leusden), Roberto Tauraso (Rome), Sep Thijssen (Nijmegen), Rohith Varma (Chennai), Traian Viteam (Montevideo) and Hans Zantema (Eindhoven).

Problem 2012-1/A Let m and n be coprime positive integers. Let Γ be the graph that has the disjoint union $\mathbb{Z}/n\mathbb{Z} \sqcup \mathbb{Z}/m\mathbb{Z}$ as vertex set and that has for every $1 \le i \le m+n-1$ an edge connecting $i \pmod n$ and $i \pmod m$. Show that Γ is a tree.

Solution We received solutions from Charles Delorme, Pieter de Groen, Alex Heinis, Ruud Jeurissen, Thijmen Krebs, Merlijn Staps, Roberto Tauraso, Sep Thijssen, Rohith Varma, Traian Viteam and Hans Zantema. The book token goes to Hans Zantema. The following solution is based on that of Sep Thijssen.
Note that Γ has $m+n$ vertices and $m+n-1$ edges, so it suffices to show that Γ is connected. Without loss of generality we may assume $n \ge m$. Then note that for all $1 \le i \le m-1$, there is an edge connecting $i \pmod m$ to $i \pmod n$ and one connecting $i \pmod n$ and $i+n \pmod m$, since $i+n \le m+n-1$. In particular, $i \pmod m$ and $i+n \pmod m$ are in the same connected component. Thus $n, 2n, \ldots, mn \pmod m$ lie in the same connected component. As n and m are coprime, this implies that $\mathbb{Z}/m\mathbb{Z}$ is inside a single connected component. Moreover, any vertex in $\mathbb{Z}/n\mathbb{Z}$ is connected to at least one of $\mathbb{Z}/m\mathbb{Z}$, hence Γ is connected.

Problem 2012-1/B Is it possible to partition a non-empty open interval in closed intervals of positive length? Let Δ be a triangle (including its interior) and let $P \in \Delta$ be an interior point. Is it possible to partition $\Delta - \{P\}$ in closed line segments of positive length?

Solution *Solution to the former question.* We received solutions from Thijmen Krebs, Merlijn Staps and Sep Thijssen. The following is based on the submission of Sep Thijssen, who receives the book token.
We will prove that it is not possible to partition a non-empty open interval (x_0, x_1) in closed intervals of positive length.
Let a partitioning $\mathcal{P}$ of (x_0, x_1) be given. We will now recursively define a sequence (x_n), starting with the given numbers x_0 and x_1. Consider the interval $[a, b]$ in $\mathcal{P}$ that contains $(x_n + x_{n+1})/2$. If n is even, we put $x_{n+2} = b$, and if n is odd, we put $x_{n+2} = a$.
The sequence $x_0, x_2, \ldots$ is increasing, and the sequence $x_1, x_3, \ldots$ is decreasing. Moreover, we have $|x_{n+1} - x_n| \le (x_1 - x_0)/2^n$, and thus the sequence $(x_n)_n$ converges. Let $I \in \mathcal{P}$ contain the limit of $(x_n)_n$. Then I contains infinitely many of the x_n, which contradicts the fact that each x_n is an endpoint of the interval of $\mathcal{P}$ it lies in.
Solution to the latter question. We received solutions from Thijmen Krebs, Paulo Perfetti and Roberto Tauraso, Merlijn Staps, Sep Thijssen and Hans Zantema.
There are many ways to partition $\Delta - \{P\}$ in closed line segments of positive length; the following figure depicts one possibility.



Problem 2012-1/C A move on a pair (a, b) of integers consists of replacing it with either $(a+b, b)$ or $(a, a+b)$. Show that starting from any pair of coprime positive integers one can obtain a pair of squares in finitely many moves.

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Oplossingen | Solutions

Solution This problem was solved by Charles Delorme, Alex Heinis, Thijmen Krebs and Hans Zantema. The book token goes to Charles Delorme. The following is based on the solution of Thijmen Krebs.

Without loss of generality we may assume that b is odd. Let p be a prime that is congruent to a modulo b and to 3 modulo 4. Such a prime exists by Dirichlet's theorem on primes in arithmetic progressions. By a finite number of moves we move from (a, b) to (p, b).

Similarly, choose a prime $q > b$ that is congruent to 3 modulo 4 and to b modulo p, and move to (p, q).

By quadratic reciprocity either p is a square modulo q or q is a square modulo p, but not both. Without loss of generality we assume that p is a square modulo q. So let x be an integer with $x^2 \equiv p$ modulo q. Let r_1 and r_2 be primes congruent to x modulo q with $r_1 \equiv 1$ and $r_2 \equiv 3$ modulo 4. Then by quadratic reciprocity, q is a square modulo either r_1 or r_2. In any case we find a prime r congruent to x modulo q such that q is a square modulo r.

Since $x^2 \equiv r^2$ modulo q, we can move to (r^2, q). As q is a square modulo r, it is also a square modulo r^2, so we may finally move to (r^2, t^2) for some t.