Problemen

| Problem Section

Edition 2011-4 We received solutions from Rik Bos, Pieter de Groen, Alex Heinis, Nicky Hekster and Merlijn Staps.

Problem 2011-4/A Let Γ be a finite undirected graph (without loops or multiple edges). Denote the set of vertices by V. Assume that there are a function $f\colon V \to \mathbf{Z}$ and a positive integer n such that

$$\sum_v \left| \sum_w (f(v) - f(w)) \right| = 2n,$$

where v runs over all the vertices of Γ and w over all the neighbours of v. Show that there are an $m \le n$ and a collection of m edges such that the graph obtained from Γ by removing those edges is not connected.

Solution We received solutions from Pieter de Groen, Alex Heinis and Merlijn Staps. The book token goes to Pieter de Groen. The following is based on the solutions of Alex Heinis and Pieter de Groen.

For a vertex $v \in V$ we write

$$\delta(v) = \sum_w f(v) - f(w),$$

where w runs over all the neighbours of v. Denote by V_+ the set of vertices v with $\delta(v) \ge 0$. Since $\sum_{v\in V} \delta(v) = 0$ and $\sum_{v\in V} |\delta(v)| = 2n$, we must have $\sum_{v\in V_+} \delta(v) = n$.

Let M be the maximal value of f. Consider the partition $V = A \cup B$ with $A = f^{-1}(M)$ and $B = V \setminus A$. Since f is not constant, we know that A and B are non-empty. Clearly A is a subset of V_+.

Let E be the set of edges connecting A and B. Clearly the group obtained by removing E from Γ is not connect. Moreover, we have

$$|E| = \sum_{e\in E} 1 \le \sum_{v\in A} \delta(v) \le \sum_{v\in V_+} \delta(v) = n,$$

where the first inequality follows because for every neighbour w of a $v \in A$ we have $f(v) - f(w) \ge 1$.

Problem 2011-4/B Let ϵ be a positive real number. Show that there is a finite group G that is not a 2-group, but in which the proportion of elements that have 2-power order is at least $1 - \epsilon$.

Solution We received solutions from Nicky Hekster, Alex Heinis and Merlijn Staps. The book token goes to Merlijn Staps. The following is based on the solution of Nicky Hekster.

Let $n > 0$ be such that $2^{-n} < \epsilon$. Let p be a prime number congruent to 1 modulo 2^n. Let C be the unique subgroup of $\mathbf{F}_p^\times$ of order 2^n. Consider the subgroup

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in C, b \in \mathbf{F}_p \right\}$$

of the group of invertible 2 by 2 matrices over $\mathbf{F}_p$. Then the order of G is $p2^n$. For every $a \in C \setminus \{1\}$ and $b \in \mathbf{F}_p$ the matrix

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

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is conjugate to the matrix

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

and thus has 2-power order. Therefore the proportion of elements of 2-power order is at least $(2^n - 1)/2^n > 1 - \epsilon$.

Problem 2011-4/C Let B be a commutative ring and A a subring of B. Assume that the additive group of A has finite index in B. Show that the unit group of A has finite index in the unit group of B.

Solution We received solutions from Rik Bos, Nicky Hekster and Alex Heinis. The book token goes to Nicky Hekster. All solutions were similar to the following.

The set $I = \{a \in A\colon aB \subset A\}$ is an ideal in both A and B. The kernel U of the homomorphism $B^\times \to (B/I)^\times$ is contained in $1 + I$, which is contained in A. Because U is a multiplicative group, we find $U \subset A^\times$, so it suffices to show that $B^\times/U$ is finite. As this injects into $(B/I)^\times$, we are reduced to showing that the latter is finite.

Note that B/A is a finite group, so the endomorphism ring $\mathrm{End}(B/A)$ is finite. Note also that the kernel of the ring homomorphism $B \to \mathrm{End}(B/A)$ that sends b to the map $(x \mapsto bx)$ is exactly I, so we get an injection $B/I \to \mathrm{End}(B/A)$. It follows that B/I is finite as well, and therefore so is $(B/I)^\times$.

Correction to Problem 2011-3/A

Problem 2011-3/A Fix a point P in the interior of a face of a regular tetrahedron Δ. Show that Δ can be partitioned in four congruent convex polyhedra such that P is a vertex of one of them.

Correction We thank Jan C. Smit (Nieuwegein) for kindly notifying us that our solution to this problem was incomplete. Indeed, referring to the published solution, the polyhedra we constructed are convex only in the case that P_d lies inside the triangle $Q_{ab}Q_{bc}Q_{ac}$. Jan Smit also gave the following fix, which is identical to some of the solutions we had received originally. Assume without loss of generality that P_d lies inside the triangle $AQ_{ab}Q_{ac}$. Then the polyhedron $AP_bP_cP_dQ_{ab}Q_{ac}Q_{ad}Z$ is not convex. The plane that contains Q_{ad}, Q_{bc}, Z, P_a and P_d cuts this polyhedron into two parts. One checks that the union of one of these parts and the image of the other part under the rotation $(AD)(BC)$ is also a convex polyhedron and the images of this union under V_4 give the desired partition.