Problemen

| Problem Section

Edition 2011-3 We received solutions from Pieter de Groen (Brussel), Alex Heinis (Hoofddorp), Tejaswi Navilarekallu (Amsterdam), Hendrik Reuvers (Maastricht) and Albert Stadler (Herrliberg).

Problem 2011-3/A Fix a point P in the interior of a face of a regular tetrahedron Δ. Show that Δ can be partitioned in four congruent convex polyhedra such that P is a vertex of one of them.

Solution We received solutions from Pieter de Groen, Alex Heinis, Tejaswi Navilarekallu and Hendrik Reuvers. The book token goes to Hendrik Reuvers.
Let A, B, C and D be the vertices of Δ. Define $\sigma_1 = (AB)(CD)$ to be the rotation of Δ that interchanges A with B and C with D. Similarly write $\sigma_2 = (AC)(BD)$ and $\sigma_3 = (AD)(BC)$. Then $V_4 = \{\mathrm{id}, \sigma_1, \sigma_2, \sigma_3\}$ is a subgroup of the symmetry group of Δ.
Let P_a, P_b, P_c and P_d be the images of P under V_4 in the faces BCD, CDA, DAB and ABC, respectively. (For example, if P lies in face ABC, then $P = P_d$.) Let Z be the orthocenter of Δ. Define Q_{ab} to be the intersection point of AB with the plane through Z, P_c and P_d, and analogously define Q_{ac}, Q_{ad}, Q_{bc}, Q_{bd} and Q_{cd}.
Now the four polyhedra

$$AP_bP_cP_dQ_{ab}Q_{ac}Q_{ad}Z,$$
$$BP_cP_dP_aQ_{bc}Q_{bd}Q_{ab}Z,$$
$$CP_dP_aP_bQ_{cd}Q_{ac}Q_{bc}Z,$$
$$DP_aP_bP_cQ_{ad}Q_{bd}Q_{cd}Z$$

cover Δ and are congruent, as they are mapped onto each other by the symmetries in V_4. The polyhedra are convex since each of them is the intersection of six half-spaces, three of which are defined by a plane containing a face of Δ and three by a plane through Z.

Problem 2011-3/B Let n be a positive integer. Show that 3^n divides the numerator of

$$\sum_{k=1}^{n} \frac{4k-1}{2k(2k-1)} 9^k.$$

Solution We received solutions from Alex Heinis, Tejaswi Navilarekallu and Albert Stadler. The book token goes to Albert Stadler.
We use the fact that if $q \in \mathbb{Q}$ has denominator not divisible by 3, then q can be reduced modulo 3^k for all positive integers k.
Since $2\log(1-3x) = \log(1-6x+9x^2)$ as complex functions, we have the following identity of power series:

$$2\sum_{k=1}^{\infty} \frac{3^k x^k}{k} = \sum_{k=1}^{\infty} \frac{3^k(2x-3x^2)^k}{k}. \qquad (1)$$

The coefficients (in x) of these power series are rational numbers. Since for all $k \geq 1$ we have $k \leq 3^k$, it follows that the denominator of $3^k/k$ is not divisible by 3, and hence that none of the coefficients of these power series have denominator divisible by 3.
In particular, we can reduce these coefficients modulo 3^n. For all $k \geq 2n$ we have that 3^n divides the numerator of $3^k/k$, so we obtain from (1) the congruence

$$\sum_{k=1}^{2n} \frac{2 \cdot 3^k x^k}{k} \equiv \sum_{k=1}^{2n} \frac{3^k(2x-3x^2)^k}{k} \pmod{3^n}.$$

By substituting $x = 1$, we obtain the following identity modulo 3^n:

Redactie:
Johan Bosman
Gabriele Dalla Torre
Jinbi Jin
Ronald van Luijk
Lenny Taelman
Wouter Zomervrucht

Problemenrubriek NAW
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

$$
\begin{aligned}
0 &\equiv \sum_{k=1}^{2n} \frac{2\cdot 3^k-(-3)^k}{k} \\
&\equiv \sum_{k=1}^{n} \frac{3^{2k}}{2k-1} + \sum_{k=1}^{n} \frac{3^{2k}}{2k} \\
&\equiv \sum_{k=1}^{n} \Big(\frac{1}{2k-1}+\frac{1}{2k}\Big)9^k \\
&\equiv \sum_{k=1}^{n} \frac{4k-1}{2k(2k-1)}9^k.
\end{aligned}
$$

Hence 3^n divides the numerator of

$$\sum_{k=1}^{n} \frac{4k-1}{2k(2k-1)}9^k,$$

as desired.

One can also prove identity (2) for $x=1$ by looking at the so-called *3-adic logarithms* of -2 and 4 on the *3-adic integers*, see for example the book *p-adic Numbers: An Introduction* by F.Q. Gouvea.

Problem 2011-3/C Let $n>1$ be an integer. Show that there are no non-linear complex polynomials $f(X)$ such that

$$f^n(X)-X=(f\circ f\circ\cdots\circ f)(X)-X$$

is divisible by $(f(X)-X)^2$.

Solution We received a solution from Alex Heinis, who wins the book token.

Assume f satisfies the condition. Put $g(X):=f(X)-X$.

We claim that

$$f^n(X)-X\equiv g(X)\left(1+f'(X)+\cdots+f'(X)^{n-1}\right)$$

modulo $g(X)^2$ for all $n\geq 1$.

Clearly the claim holds for $n=1$. Assume it holds for $n=N-1$. Using that for all polynomials a we have

$$f(X+g(X)a)\equiv f(X)+g(X)af'(X) \pmod{g(X)^2},$$

we find

$$
\begin{aligned}
f^N(X)-X &\equiv f(f^{N-1}(X)-X+X)-X \\
&\equiv f(X+g(X)(1+f'(X)+\cdots+f'(X)^{N-2}))-X \\
&\equiv g(X)+f'(X)g(X)(1+f'(X)+\cdots+f'(X)^{N-2}) \\
&\equiv g(X)(1+f'(X)+\cdots+f'(X)^{N-1}),
\end{aligned}
$$

which proves the claim.

Since $g(X)^2$ divides $f^n(X)-X$ we find that $g(X)$ divides

$$1+f'(X)+\cdots+f'(X)^{n-1}=\frac{f'(X)^n-1}{f'(X)-1}.$$

Let $x\in\mathbb{C}$ be a root of $g(X)$. From the above equation we find that $f'(x)^n=1$. We claim that $f'(x)\neq 1$, and in particular that x is a simple root of $g(X)$. To see this, assume that $f'(x)=1$.

Let k be the multiplicity of x as a root of $f'(X)-1$. Then x must be a root of multiplicity $k+1$ of $f'(X)^n-1$, and hence a k-fold root of its derivative $nf'(X)^{n-1}f''(X)$. But since $f'(x)=1$ it follows that the multiplicity of x as a root of $f''(X)$ is k, a contradiction.

Now assume that f is not linear. Clearly f cannot be constant, so the degree of f is at least 2, and hence also the degree of g is at least 2. But then the residue theorem gives

$$0=\sum_{g(x)=0}\frac{1}{g'(x)}=\sum_{g(x)=0}\frac{1}{f'(x)-1},$$

where the sums range over all roots x of $g(X)$. Since $f'(x)^n=1$ for all such x, we have that all the $f'(x)$ lie on the unit circle. In particular, the real part of the right-hand side is strictly negative, a contradiction.