Problemen

| Problem Section

Edition 2011-2 We have received correct solutions from Pieter de Groen (Brussel), Alex Heinis (Hoofddorp), Wim Hesselink (Groningen), Alexander van Hoorn (Abcoude), Thijmen Krebs (Nootdorp), Tejaswi Navilarekallu (Amsterdam), Michiel Smid (Ottawa), Rob van der Waall (Huizen) and Martijn Weterings (Wageningen).

Problem 2011-2/A Let S be an open subset of $\mathbb{R}_{>0}$ that contains arbitrarily small elements. Prove that every positive real number can be written as a sum of finitely many elements of S.

Solution We have received solutions from Pieter de Groen, Alex Heinis, Wim Hesselink, Alexander van Hoorn, Thijmen Krebs, Tejaswi Navilarekallu, Michiel Smid and Martijn Weterings. The book token goes to Wim Hesselink.

Let x be a positive real number. By assumption there exist $y < x$ and $0 < \epsilon < x - y$ such that the open interval $(y - \epsilon, y + \epsilon)$ is contained in S. Also choose $z \in S$ with $z < 2\epsilon$. Then the interval $(x - y - \epsilon, x - y + \epsilon)$ contains a multiple nz of z, with $n \geq 0$ an integer. Now $w = x - nz$ lies in $(y - \epsilon, y + \epsilon) \subset S$ and $x = w + nz$ is a finite sum of elements of S.

Problem 2011-2/B Let n be a positive integer. Show that every sequence of n elements of $\{0, \dots, 9\}$ occurs as a sequence of consecutive digits in the last $2n$ digits of the decimal representation of some power of 2. Also, determine all $\alpha \in \mathbb{R}_{>0}$ for which the statement still holds if we replace $2n$ by $\lceil \alpha n \rceil$.

Solution We have received solutions from Alex Heinis, Tijmen Krebs and Tejaswi Navilarekallu. The book token goes to Alex Heinis. The following is partly based on the solution of Tejaswi Navilarekallu.

Note that for every positive integer n, 2 is a primitive root mod 5^n. We will use this fact later on. We prove that the statement holds if and only if $\alpha \geq \frac{\log 10}{\log 5}$. First note that for $\alpha < 1$, the statement cannot hold, as $\lceil \alpha n \rceil < n$ for sufficiently large n.

So suppose $1 \leq \alpha < \frac{\log 10}{\log 5}$. Note that there are at most $4 \cdot 5^{\lceil \alpha n \rceil - 1}$ residue classes modulo $5^{\lceil \alpha n \rceil}$ that are residues of powers of 2. Also note that modulo $2^{\lceil \alpha n \rceil}$, there are $\lceil \alpha n \rceil$ residue classes that are residues of powers of 2, and that for $k \geq \lceil \alpha n \rceil$, we have $2^k \equiv 0 \pmod{2^{\lceil \alpha n \rceil}}$. Hence, by the Chinese remainder theorem, there are at most $A_n = 4 \cdot 5^{\lceil \alpha n \rceil - 1} + \lceil \alpha n \rceil$ residue classes modulo $10^{\lceil \alpha n \rceil}$ that are residues of powers of 2.

Each of these A_n residue classes gives rise to at most $\lceil \alpha n \rceil - n + 1$ sequences of n digits in the last $\lceil \alpha n \rceil$ digits of a power of 2. So if we can show that there exists an n with $(\lceil \alpha n \rceil - n + 1) A_n < 10^n$, then we are done. To this end, note that the left-hand side can be bounded from above by a constant multiple of $n 5^{\alpha n}$. Since $5^\alpha < 10$, for any $c \in \mathbb{R}_{>0}$, we have for sufficiently large n that $cn5^{\alpha n} < 10^n$. Hence also $(\lceil \alpha n \rceil - n + 1) A_n < 10^n$ for sufficiently large n.

Now suppose $\alpha \geq \frac{\log 10}{\log 5}$. Let $n \geq 1$ be an integer. Suppose $a_0, \dots, a_{n-1} \in \{0, 1, \dots, 9\}$ and set $A = \sum_{k=0}^{n-1} 10^k a_k$. Note that for $r = \lceil \alpha n \rceil - n$, we have $2^{n+r} < 10^r$. Therefore, there is a multiple Y of 2^{n+r} with $A \cdot 10^r < Y < (A+1) \cdot 10^r$. So let $s \leq r$ be the smallest positive integer for which there exists a multiple Y of 2^{n+s} with $A \cdot 10^s < Y < (A+1) \cdot 10^s$.

If Y were divisible by 10, then it would follow that $s \geq 2$, as there are no multiples of 10 between $10A$ and $10(A+1)$. Then $A \cdot 10^{s-1} < \frac{Y}{10} < (A+1) \cdot 10^{s-1}$, and $\frac{Y}{10}$ is divisible by 2^{n+s-1}, contradicting the minimality of s. Hence Y is coprime to 5. Since 2 is a primitive root mod 5^n, it follows that there exists a $t \geq n + s$ with $2^t \equiv Y \pmod{5^{n+s}}$. We also have $2^t \equiv 0 \equiv Y \pmod{2^{n+s}}$, so by the Chinese remainder theorem, it follows that $2^t \equiv Y \pmod{10^{n+s}}$. In other words, the last $n + s$ digits of 2^t are the same as those of Y. In particular, 2^t contains A as a sequence of digits in its last $\lceil \alpha n \rceil$ digits.

Problem 2011-2/C Let p be a prime number. Determine the smallest integer d for which there is a monic polynomial f of degree d with integer coefficients such that p^{p+1} divides $f(n)$ for all integers n.

Solution We have received solutions from Pieter de Groen, Alex Heinis, Thijmen Krebs, Tejaswi

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Oplossingen | Solutions

Navilarekallu, Michiel Smid, Rob van der Waall and Martijn Weterings. The book token goes to Tejaswi Navilarekallu, whose solution was as follows.

We claim that the smallest degree is p^2.

For a polynomial $f(X) \in \mathbb{Z}[X]$ we denote by $f^{(1)}$ the polynomial $f(X+1) - f(X)$ and by $f^{(m)}$ the polynomial obtained by repeating this operation m times. Now assume that f is monic of degree d, and that p^{p+1} divides $f(n)$ for all n. Then also p^{p+1} divides $f^{(d)}(n)$ for all n. But we have $f^{(d)}(X) = d!$, so we conclude that p^{p+1} divides $d!$ and hence that d is at least p^2.

To see that the minimal degree is exactly p^2, it suffices to observe that the polynomial

$$f(X) = (X-1)(X-2)\cdots(X-p^2)$$

satisfies the required condition.