Problemen

| Problem Section

Edition 2009-4 We received submissions from Pieter de Groen (Brussel) and Thijmen Krebs (Nootdorp).

Problem 2009-4/A Is there a polynomial with rational coefficients whose minimum on the real line is $\sqrt{2}$?

Solution This problem was solved by Pieter de Groen and Thijmen Krebs. Pieter de Groen receives the book token.
We will show that Krebs' polynomial $f(x) = g(x^2)$ with

$$g(x) = \tfrac{1}{8}(3x^4 - 2x^3 - 12x^2 + 12x + 12)$$

is such a polynomial. Indeed, note that the derivative of g satisfies $4g'(x) = 3(x^2 - 2)(2x-1)$, so that $g(x)$ has local minima at $x = \pm\sqrt{2}$ and a local maximum at $x = \frac{1}{2}$. From $g(0) > g(\sqrt{2})$ we conclude that the minimum of g on the interval $[0, \infty)$ equals $g(\sqrt{2}) = \sqrt{2}$. It follows that the minimum of $f(x) = g(x^2)$ on the real line equals $\sqrt{2}$ as well. ∎

Problem 2009-4/B Are there infinitely many positive integers whose positive divisors sum to a square?

Solution Suppose there are only finitely many such integers and let $N > 1$ be a common multiple. For any $x \in \mathbf{R}$, let $S(x)$ denote the set of all prime powers p^r with $p \le x$ prime and $r \ge 1$ the smallest integer for which p^r does not divide N.
For every integer n, let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of n. The function σ is weak multiplicative, meaning that $\sigma(mn) = \sigma(m)\sigma(n)$ whenever m and n are coprime. Let q be any prime larger than $\sigma(m)$ for all $m \in S(N)$ and let t denote the number of primes up to and including q. For all t prime powers $m = p^r \in S(q)$ with p prime, the prime divisors of $\sigma(m)$ are smaller than q; for $p \le N$ this follows by definition of q, while for $p > N$ it follows from the fact that $\sigma(m) = p + 1$ is even, so all its prime divisors are at most $(p+1)/2 < q$.
We conclude that the $\mathbf{F}_2$-subspace of $\mathbf{Q}^*/\mathbf{Q}^{*2}$ generated by the elements $\sigma(m)$ for all $m \in S(q)$ is contained in the subspace generated by all primes smaller than q, which has dimension $t-1$. This implies that the t elements $\sigma(m)$ for $m \in S(q)$ are linearly dependent, so there exists a nonempty subset $T \subset S(q)$ such that for $n = \prod_{m\in T} m$ the weak multiplicativity of σ yields $\sigma(n) = \prod_{m\in T}\sigma(m) = 1 \in \mathbf{Q}/\mathbf{Q}^{*2}$. Therefore $\sigma(n)$ is a square, which contradicts the fact that n is not a divisor of N. This proves that there are infinitely many integers whose divisors sum to a square. ∎

Problem 2009-4/C For which odd positive integers n do there exist an odd integer $k > n$ and a subset $S \subset \mathbf{Z}/k\mathbf{Z}$ of size n such that for every non-zero element $r \in \mathbf{Z}/k\mathbf{Z}$ the cardinality of the intersection $S \cap (S + r)$ is even? What about even n?

Solution This problem was solved by Thijmen Krebs, who receives the book token. The following solution of the problem is based on his solution.
We claim that for $n \equiv 2 \bmod 4$ and for $n \equiv 3 \bmod 4$ there does not exist any odd integer k with the requested property, whereas for $n \equiv 0 \bmod 4$ and for $n \equiv 1 \bmod 4$ there exists such an odd integer k.
We denote by $\bar{r}$ the residue class of the integer r in $\mathbf{Z}/k\mathbf{Z}$.
Firstly, we suppose $n \equiv 2 \bmod 4$ or $n \equiv 3 \bmod 4$. For any candidates k and S and any $r \in \mathbf{Z} \setminus k\mathbf{Z}$ we have

$$|S \cap (S + \bar{r})| = |S \cap (S - \bar{r})| \equiv 0 \bmod 2.$$

By summing over all $r \in 1, \ldots, k-1$ we get

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$$n(n-1) = \sum_{r=1}^{k-1} |S \cap (S+\bar{r})| = \sum_{r=1}^{(k-1)/2} (|S \cap (S+\bar{r})| + |S \cap (S-\bar{r})|) \equiv 0 \bmod 4.$$

This proves the first part of our claim.
Now we will show that for $n \equiv 0 \bmod 4$ and for $n \equiv 1 \bmod 4$ there exists such an odd integer k. Let A be the set $\{0,1,2,4\}$. If $n=4$ then it is easy to check by hand that for $k=7$ the subset $\bar{A}$ of $\mathbf{Z}/7\mathbf{Z}$ satisfies the requested property.
If $n \equiv 0 \bmod 4$ we pick any odd integer h greater than $n/4$ and any subset $B \subseteq \{0,\dots,h-1\}$ of cardinality $n/4$. We claim that we can choose $k=7h$ and $S = \{ha+b \bmod 7h : a \in A, b \in B\}$. For any $r \in \mathbf{Z}/7h\mathbf{Z}$ we have

$$|S \cap (S+r)| = \sum_{(b,c)\in B\times B} \left| (\overline{hA}) \cap (\overline{hA+r+b-c}) \right| =$$

$$= \sum_{\substack{(b,c)\in B\times B\\ r\equiv c-b \bmod h}} \left| (\overline{hA}) \cap \left(\overline{hA + h\frac{(r+b-c)}{h}} \right) \right|$$

All the terms in the last sum are even and equal to either 2 or 4, because it is equal to what we have computed in the case $n=4$ and $k=7$, namely the cardinality of the intersection between the sets $\overline{A}$ and $\overline{A+\frac{(r+b-c)}{h}}$ in $\mathbf{Z}/7\mathbf{Z}$.
Now we suppose $n \equiv 1 \bmod 4$ and let A and B be the sets $\{0,\dots,(n-1)/2\}$ and $\{1,2\}$, respectively. We claim that we can choose $k = \frac{3(n+1)}{2}$ and $S = \{3a+b \bmod k : a \in A, b \in B\} \setminus \{1 \bmod k\}$. For any $r \in \mathbf{Z} \setminus k\mathbf{Z}$ we have

$$|S \cap (S+\bar{r})| = \sum_{(b,c)\in B\times B} \left| \overline{3A} \cap (\overline{3A+r+b-c}) \right| - \left| S \cap (\overline{1+r}) \right| - \left| S \cap (\overline{1-r}) \right|.$$

Note that the sum $\sum_{(b,c)\in B\times B} \left| \overline{3A} \cap (\overline{3A+r+b-c}) \right|$ is equal to $(n+1)$ if $r \equiv 0 \bmod 3$ and to $(n+1)/2$ if $r \not\equiv 0 \bmod 3$. We can conclude by observing that in the first case $\overline{1+r} \in S$ if and only if $\overline{1-r} \in S$ and in the second case $\overline{1+r} \in S$ if and only if $\overline{1-r} \notin S$. ■