Problemen

| Problem Section

Edition 2009-3 We received submissions from Daniël Worm (Leiden), Rutger Kuyper (Nijmegen), Thijmen Krebs (Nootdorp), Jaap Spies (Emmen), Sander Scholtus (Den Haag), Pieter de Groen (Brussel), Dan Dima (Bucharest), Wim Schikhof (Nijmegen), and Sep Thijssen (Nijmegen).

Problem 2009-3/A Let k be a non-negative integer. Let $S \subset \mathbf{Z}$ be a set consisting of $2^{k+1}-1$ integers. Show there exists a subset $T \subset S$ of cardinality 2^k such that the sum of the elements of T is divisible by 2^k.

Solution This problem was solved by Daniël Worm, Thijmen Krebs, Rutger Kuyper, Sander Scholtus, and Sep Thijssen. All submitted essentially the same solution. The book token goes to Rutger Kuyper.

We use induction. For $k=0$ the statement clearly holds. Suppose the statement holds for some non-negative integer k. Let $S \subset \mathbf{Z}$ be a subset of cardinality $2^{k+2}-1$. Using the induction hypothesis on two disjoints subsets of S of cardinality $2^{k+1}-1$ each, we can find disjoint subsets T_1 and T_2 of S of cardinality 2^k, such that the sum of their elements is divisible by 2^k. Now note that the complement of $T_1 \cup T_2$ in S has cardinality $2^{k+1}-1$, so by using the induction hypothesis once more we find a subset $T_3 \subset S$, disjoint with T_1 and T_2, of cardinality 2^k and such that the sum of its elements is divisible by 2^k. To conclude, choose $i \neq j$ such that T_i and T_j have the same sum modulo 2^{k+1} and observe that $T = T_i \cup T_j \subset S$ satisfies the requirements.

Problem 2009-3/B Find all functions $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that

$$f(x+y) \geq f(x) + yf(f(x)) \tag{1}$$

for all x and y in $\mathbf{R}_{>0}$.

Solution This problem was solved by Dan Dima, Pieter de Groen, Thijmen Krebs, Sep Thijssen, and Daniël Worm. The book token goes to Daniël Worm.

The following is essentially the solution by Thijmen Krebs and Daniël Worm.

Suppose such a function f exists. For all $x, y > 0$ we have $f(x+y) > f(x)$, so f is strictly increasing. For fixed x, the right-hand side of (1) is linear in y, so f is unbounded. Therefore, we may choose an $x > 0$ such that $f(f(x)) > 1$ and a $y > 0$ satisfying $y(f(f(x)) - 1) > x + 1$. Then for $z = x + y$ we have

$$f(z) = f(x+y) \geq f(x) + yf(f(x)) > yf(f(x)) > x + y + 1 = z + 1.$$

However, from $f(z+1) \geq f(z) + f(f(z)) > f(f(z))$ and the fact that f is increasing, we find $z + 1 > f(z)$. From this contradiction we conclude that no such f exists.

Problem 2009-3/C Let V be an infinite-dimensional vector space. Show that the dimension of the dual space V^* equals the cardinality of V^*.

Solution We received no solutions to this problem. Wim Schikhof pointed out that the solution can be found in the literature (G. Köthe, *Topologische Lineare Räume I*, 1960), where it is known as a theorem of Erdős and Kaplansky. Bas Edixhoven communicated the following (folklore) proof.

We denote the cardinality of a set S by $|S|$.

Let V be an infinite-dimensional vector space over a field k. Clearly $|V^*| \geq \dim(V^*)$, so we only need to show $|V^*| \leq \dim(V^*)$.

Choose a basis I of V, using Zorn's Lemma. Let k^I be the set of all functions from I to k and let $k[I]$ be the vector space of all polynomials in the elements of I. Consider the

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'evaluation' map

$$e : k^I \to (k[I])^* : f \mapsto \Big[P \mapsto P\big(f(i)_{i\in I}\big)\Big].$$

We claim that the images of e are linearly independent. To see this, let $f_1, \dots, f_n$ be distinct elements of k^I and let

$$\alpha_1 e(f_1) + \cdots + \alpha_n e(f_n) = 0$$

be a linear relation amongst their images. Note that there is a finite subset $J \subset I$ on which the functions f are already distinct. In particular, for any $1 \le j \le n$ we can choose a polynomial $P \in k[J] \subset k[I]$ that evaluates to 1 on f_j and to 0 on all the other f's, so that

$$\alpha_j = \alpha_1 P\big(f_1(i)_{i\in I}\big) + \cdots + \alpha_n P\big(f_n(i)_{i\in I}\big) = 0,$$

which proves the claim.
Now, as the images of e are linearly independent we have

$$|k^I| \le \dim(k[I]^*).$$

But k^I is isomorphic to V^* and $k[I]$ is isomorphic to V (since I has the same cardinality as the set of all monomials in I), so we conclude $|V^*| \le (\dim V^*)$.