Problemen

| Problem Section

Edition 2009-2 We received solutions from Rob van der Waall (Huizen), Thijmen Krebs (Nootdorp), Ruud Jeurissen (Nijmegen), Tejaswi Navilarekallu (Amsterdam), Hendrik Lenstra (Leiden), and Jaap Spies (Emmen).

Problem 2009-2/A In how many ways can one place coins on an $n \times n$ chessboard such that for every square the number of (horizontally or vertically) adjacent squares that contain a coin is odd?

Solution We received solutions from Tejaswi Navilarekallu and Thijmen Krebs. Tejaswi Navilarekallu will receive the book token.
We contend that the required task is impossible if n is odd and that it can be performed in exactly 2^n ways if n is even.
We use integral coordinates (i, j) with $1 \leq i, j \leq n$ for the squares of the board. We color a square (i, j) white if $i + j$ is even, and black if it is odd. The problems of placing coins on the white and on the black squares are independent. We say that a configuration of coins is legal at a square if that square has an odd number of neighbors containing a coin. First we treat the case of even n, so $n = 2k$ for some integer k.
Claim: Any configuration of coins on the white half-diagonal $(1, 1)$, $(2, 2)$, …, (k, k) extends to a unique configuration on all the white squares that is legal at all black squares.
Proof. We show by induction on m that any configuration on the half-diagonal extends uniquely to a configuration on the white squares (i, j) with $i + j \leq 2m$ that is legal at all black squares (i, j) with $i + j < 2m$. Note that the uniqueness implies that this configuration will be symmetric in the sense that there is a coin on (i, j) if and only if there is a coin on (j, i).
The case $m = 1$ is trivial. For $m \leq n$ the induction step is easy, working outwards from the half-diagonal. For $m > n$ we work from the squares $(n, 2m - n)$ and $(2m - n, n)$ on the edge towards the square (m, m) on the diagonal. The symmetry guarantees that there is no conflict at the square (m, m). □
It follows that the number of legal configurations on the white squares is 2^k. Of course the same holds for the black squares, so the total number of legal configurations on the board is $2^k \cdot 2^k = 2^n$.
We shall now prove by contradiction that there are no legal configurations if n is odd. Let k be such that $n = 2k + 1$. Assume that we are given a legal configuration. Legality at the corner square $(1, 1)$ implies that exactly one of $(1, 2)$ and $(2, 1)$ contains a coin. Legality at $(2, 2)$ then implies that either both or none of $(2, 3)$ and $(3, 2)$ have a coin. Continuing this alternating process we see that for all i either both or none of $(2i, 2i + 1)$ and $(2i + 1, 2i)$ have a coin. In particular either both or none of $(n - 1, n)$ and $(n, n - 1)$ have a coin, which contradicts legality at the corner square (n, n).

Problem 2009-2/B A magic $n \times n$ matrix of order r is an $n \times n$ matrix whose entries are non-negative integers and whose row and column sums all equal r. Let $r > 0$ be an integer. Show that a magic $n \times n$ matrix of order r is the sum of r magic $n \times n$ matrices of order 1.

Solution This problem was solved by Ruud Jeurissen, Thijmen Krebs, Tejaswi Navilarekallu, and Jaaps Spies. The following is essentially the solution by Ruud Jeurissen, which was similar to all others. Ruud Jeurissen is the winner of the book token.
We prove the statement by induction on r, the case $r = 1$ being trivial. Suppose M is a magic $n \times n$ matrix of order r. We associate to M the bipartite graph where the two underlying sets R and C of vertices consist of the rows and columns of M respectively, and the i-th row and j-th column are connected by M_{ij} edges. Each subset $S \subset R$ of size k is connected by kr edges to columns in C. Since each column in C has only r edges, this implies that there are at least k columns in C that are connected to S. By Hall's theorem (also known as the marriage theorem), this implies there is a matching from R to C, meaning there is a magic matrix M' of order 1, such that the entries of

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$M'' = M - M'$ are non-negative. This implies that M'' is a magic matrix of order $r-1$, so by the induction hypothesis M'' is a sum of $r-1$ magic matrices of order 1. We conclude that $M = M'' + M'$ is the sum of r magic matrices of order 1.

Problem 2009-2/C (proposed by Tejaswi Navilarekallu) Find all finite groups G with the property that for all $g, h \in G$ at least one of (g,h), (g,gh) and (h,hg) is a pair of conjugate elements.

Solution We received solutions from Rob van der Waall and Hendrik Lenstra. The proposer and Hendrik Lenstra had similar solutions, and the following is based on both. Hendrik Lenstra will receive the book token.
We claim that the only groups satisfying the given condition are $\{1\}$, $\mathbf{Z}/2\mathbf{Z}$, and the dihedral groups of order 6 and 10.
Clearly the trivial group and the group of order two satisfy the condition, so assume G has $n > 2$ elements.
Let $1 = d_1, d_2, \ldots, d_k$ be the sizes of the conjugacy classes of G. Therefore, $d_1 + \cdots + d_k = n$. Then the set

$$\{(g,h) \in G \times G | g, h \text{ are conjugates}\}$$

has precisely $d_1^2 + d_2^2 + \cdots + d_k^2$ elements. Similary, the sets

$$\{(g,h) \in G \times G | g, gh \text{ are conjugates}\}$$

and

$$\{(g,h) \in G \times G | h, hg \text{ are conjugates}\}$$

have exactly $d_1^2 + \cdots + d_k^2$ elements. Note that $(1,1)$ belongs to all three sets. For G to satisfy the condition in the problem, we need the union of the above three sets to be $G \times G$. In particular, this gives the inequality $3(d_1^2 + \ldots + d_k^2) \geq n^2 + 2$ or equivalently,

$$3(d_2^2 + \cdots + d_k^2) \geq n^2 - 1. \tag{1}$$

Note that the d_i divide n. If for all i we have $d_i \leq n/3$ then

$$3(d_2^2 + \cdots + d_k^2) \leq 3(d_2 + \cdots + d_k)\frac{n}{3} = (n-1)n$$

contradicting the inequality (1). So there must be a conjugacy class C with exactly $n/2$ elements. We are going to show that, except for the identity, all other conjugacy classes have exactly 2 elements.
Since the conjugation action of G on C is transitive the centralizer of an element $c \in C$ has at most two elements, and since that centralizer contains c it follows that $c^2 = 1$, so all elements of C have order 2.
Let a and b be elements of C. We will show by contradiction that ab is not in C. Assume that ab is in C. Then $abab = 1$, so $aba^{-1} = b$, so $a = b$, since the stabilizer of b consists of only 1 and b. We conclude that $ab = 1$, a contradiction.
Next we show that the complement $H = G - C$ is a subgroup. Let x and y in H be given and fix an $r \in C$. By the above x and y can be written as ar and rb respectively, with $a, b \in C$. Then xy equals ab, which is an element of H.
The action by conjugation of an element $c \in C$ on H is an involution, since $c^2 = 1$. The only fixed point of this action is $1 \in H$, because $chc^{-1} = h$ implies $h^{-1}ch = c$ and the centralizer of c is $\{1, c\}$.
Any finite group H with an involution σ that fixes only $1 \in H$ is necessarily abelian, and the involution must be inversion. To see this, first observe that by a counting argument, every element $x \in H$ can be written as $x = \sigma(h)h^{-1}$ for some $h \in H$, then apply σ to

obtain $\sigma(x) = x^{-1}$. Hence the automorphism is inversion, and therefore the group is abelian.

So we may assume $d_1 = 1$, $d_2 = n/2$, and $d_3 = \cdots = d_k = 2$. Together with the inequality (1) this implies that $n \leq 10$. It is now easy to check that only the dihedral groups of order 6 and 10 satisfy the required condition (with $n > 2$.)