Edition 2009-1 We received solutions from Tejaswi Navilarekallu (Amsterdam), Ladislav Matejíčka (Bratislava), Alex Heinis (Hoofddorp), Thijmen Krebs (Nootdorp), Louis Maassen (Milsbeek), Pieter de Groen (Brussel), studentenvereniging PRIME (Gent), Floor van Lamoen (Goes), Johan de Ruiter (Leiden), Sep Thijssen (Nijmegen), Noud Aldenhoven (Nijmegen), and Jaap Spies.

Problem 2009-1/A (folklore) Let s be a real number. Find all continuous functions $f\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ that satisfy

$$f(xy) = f(x)^{y^s} f(y)^{x^s}$$

for all x and y.

Solution We received correct solutions from Tejaswi Navilarekallu, Ladislav Matejíčka, Alex Heinis, Thijmen Krebs, Louis Maassen, and Pieter de Groen. All received solutions were similar to one another and to the one we present here. The book token goes to Louis Maassen (by random draw).

First we prove the following claim: if $g : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ is a continuous functions satisfying $g(xy) = g(x)g(y)$ for all x and y then there exists a real number c such that $g(x) = x^c$. Indeed, let $c \in \mathbf{R}$ be such that $g(2) = 2^c$. The multiplicativity of g then implies that $g(2^r) = (2^c)^r = (2^r)^c$ for all rational numbers r. Using the continuity of g one concludes that $g(x) = x^c$ for all positive real numbers x, which proves the claim.

Now consider the function

$$g : \mathbf{R}_{>0} \to \mathbf{R}_{>0} : x \mapsto \exp\left(x^{-s} \log(f(x))\right).$$

One verifies that $g(xy) = g(x)g(y)$ for all x and y, so by the above claim it follows that $g(x) = x^c$ for some real number c. This shows that f is necessarily of the form

$$f(x) = x^{cx^s},$$

and conversely, for every c this function satisfies the required conditions.

Problem 2009-1/B (proposed by Lee Sallows) In the accompanying picture, nine numbered counters occupy the cells of a 3×3 board so as to make a magic square. They form 8 collinear triples, and each triple yields the same sum 15.

8	1	6
3	5	7
4	9	2

Place nine counters, numbered 1 through 9, on the same board, again one in each cell, so that they form 8 new collinear triples, but now showing a common sum of 16 rather than 15.

Solution This problem was solved by Thijmen Krebs, PRIME, Floor van Lamoen, Johan de Ruiter, Sep Thijssen, Noud Aldenhoven, and Tejaswi Navilarekallu. Several others

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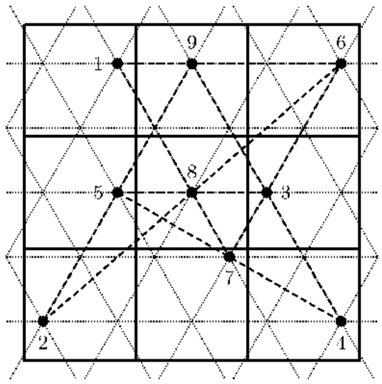
claimed to be able to prove the proposed task was impossible. The key point was to realize that the counters are not required to be placed in the center of their squares. Thijmen Krebs — who receives the book token for his solution — solved the problem by transposing a hexagonal lattice on a square lattice:
In the complex plane, let the vertices of the center square be $\frac{3}{4}(\pm 1 \pm i)$. Then for small enough counters, we can place the counters on the hexagonal lattice generated by 1 and $e^{\pi i/3}$ as follows:



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Problem 2009-1/C (folklore) If n is a non-negative integer define $a(n)$ to be the number of decimal digits of 2^n that are larger than or equal to 5. For example $a(8) = 2$. Evaluate the infinite sum

$$\sum_{i=0}^{\infty} \frac{a(n)}{2^n}.$$

Solution This problem was solved by Tejaswi Navilarekallu, Sep Thijssen, and Thijmen Krebs. Benne de Weger and Jaap Spies confirmed to us that this problem is indeed folklore and has previously appeared in several problem sections including our own! The following elegant solution is by Sep Thijssen, who receives the book token for his answer.

For a natural number m, let $s(m)$ denote the sum of all decimal digits of m; keeping track of the carries in the addition of m to itself, we find $2s(m) - s(2m) = 9b(m)$, where $b(m)$ is the number of digits of m that are at least 5. As $a(n) = b(2^n)$ by definition, we get

$$\sum_{n=0}^{\infty} \frac{a(n)}{2^n} = \frac{1}{9} \sum_{n=0}^{\infty} \frac{2s(2^n) - s(2^{n+1})}{2^n} = \frac{2}{9}s(1) - \lim_{n\to\infty} \frac{s(2^{n+1})}{9 \cdot 2^n} = \frac{2}{9},$$

where the second equality follows from the fact that the second expression can be viewed as a telescoping series.

Using a similar argument Sep Thijssen shows that for all positive real numbers $x \in [0, 10)$ we have that

$$\sum_{n=0}^{\infty} \frac{b(\lfloor 2^n x \rfloor)}{2^n} = \frac{2x}{9},$$

where $\lfloor z \rfloor$ denotes the largest integer smaller than or equal to z.