Problemen

| Problem Section

Edition 2008/4

We received submissions from Marco Pauw (London), Rob van der Waall (Huizen), Paolo Perfetti (Rome), Sep Thijssen (Nijmegen), Pieter de Groen (Brussels), Ronald Rietman (Eindhoven), Ludo Tolhuizen (Eindhoven), John Simons (Roden), Sander Scholtus (Den Haag), Kee-Wai Lau (Hong Kong), Jaap Spies, Thijmen Krebs (Delft).
In the last issue we forgot to mention Thijmen Krebs' and John Simons' correct solutions to problem 2008-3/A, Sep Thijssen's correct solution to problem 2008-3/B, and the fact that Rob van der Waall's solution to problem 2008-3/A was co-authored with Alexa van der Waall and Nils Bruin.

Problem 2008-4/A (folklore) If x is a real number then we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the largest integer smaller than or equal to x and the smallest integer bigger than or equal to x, respectively. Prove or disprove: for all positive integers n we have

$$\left\lceil \frac{2}{2^{1/n}-1} \right\rceil = \left\lfloor \frac{2n}{\log(2)} \right\rfloor .$$

Solution This problem was solved by Sep Thijssen and by Ronald Rietman & Ludo Tolhuizen. Both found the counterexample

$$n = 777451915729368$$

and used the following method to find this n. For large n the difference

$$\frac{2n}{\log(2)} - \frac{2}{2^{1/n}-1}$$

is very close to 1, so that a counterexample must necessarily have $2n/\log(2)$ very close to an integer, say m. In particular, the rational number m/n must be a good approximation to $2/\log(2)$. One can use the continued fraction expansion of $2/\log(2)$ to find rational numbers that are good approximations of $2/\log(2)$. The 36-th convergent has 777451915729368 as denominator and yields the above counterexample.

Problem 2008-4/B (folklore) Let (a_i) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$$

for some real number a. Show that

$$\lim_{n\to\infty} \frac{a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n}{n^2} = \frac{a^2}{2}.$$

Solution This problem was solved by Ludo Tolhuizen, Rob van der Waall, Pieter de Groen, John Simons, Marco Pauw, Paolo Perfetti, Sep Thijssen, Sander Scholtus, Kee-Wai Lau, Jaap Spies, and Thijmen Krebs.
The following solution is based on several of the submissions.
Put $s_n = \sum_{i=1}^n a_i$. Since we have

$$\frac{a_n}{n} = \frac{s_n}{n} - \frac{n-1}{n} \cdot \frac{s_{n-1}}{n-1}$$

and since both terms of the right-hand side tend to a, we find $\lim_{n\to\infty} a_n/n = 0$. We claim that in fact

$$\lim_{n\to\infty} \frac{\max_{i\le n} a_i}{n} = 0.$$

If the sequence (a_i) is bounded, this is obvious. If it is not, then for every n we choose $j(n) \le n$ such that $a_{j(n)} = \max_{i\le n} a_i$; clearly we have

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Oplossingen | Solutions

$$\frac{\max_{i\leq n} a_i}{n} \leq \frac{a_{j(n)}}{j(n)}$$

and the right-hand side tends to zero because the index $j(n)$ tends to infinity, while a_j/j tends to zero when j goes to infinity.
For all n we have

$$\frac{s_n^2}{n^2} - 2\cdot\frac{a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n}{n^2} = \frac{a_1^2 + \cdots + a_n^2}{n^2} \leq \frac{\max_{i\leq n} a_i}{n}\cdot\frac{s_n}{n}.$$

The first factor of the right-hand side tends to 0 while the second factor tends to a, so the product tends to 0. We conclude that

$$\lim_{n\to\infty}\frac{a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n}{n^2} = \lim_{n\to\infty}\frac{s_n^2}{2n^2} = \frac{a^2}{2}.$$

Problem 2008-4/C (proposed by Hendrik Lenstra) Let x be a real number, and m and n positive integers. Show that there exist polynomials f and g in two variables and with integer coefficients, such that

$$x = \frac{f(x^n, (1-x)^m)}{g(x^n, (1-x)^m)}.$$

Solution Unfortunately we received no submissions. The following is the proposer's solution.

The existence of the requested polynomials f and g is equivalent with the fact that x is contained in the field $K = \mathbf{Q}(x^n, (1-x)^m)$. Suppose x is not contained in K and let F be the minimal polynomial of x over K. Then F has a root $y \in \mathbf{C}$ with $y \neq x$. Every polynomial with coefficients in K of which x is a root, is a multiple of F, and therefore also has y as a root. Applying this to the polynomials $(1-T)^m - (1-x)^m$ and $T^n - x^n$, we find $(1-y)^m = (1-x)^m$ and $y^n = x^n$, from which we conclude $|y-1| = |x-1|$ and $|y| = |x|$. The circles in $\mathbf{C}$ centered at 1 and 0 with radii $|x-1|$ and $|x|$ intersect in two conjugate points or a unique real point, if the circles intersect at all. Since y and the real number x are intersection points, we find $y = x$, which is a contradiction from which we conclude $x \in K$.