Problemen

| Problem Section

Star problems. In the June 2008 edition of the NAW we revisited a selection of unsolved star problems. Whoever sent in a solution first before July 1, 2009 would receive a book token. In this and upcoming editions we will publish some of the solutions we have received.

Problem (Star) 2008-2/3 Let A and B be $n \times n$ matrices over $\mathbf{C}$. Suppose that $\lim_{k\to\infty}(A^k + B^k)$ exists. Show that there exists $M \in \mathbf{C}^{n\times n}$ such that $\lim_{k\to\infty} A^k - kM$ and $\lim_{k\to\infty} B^k + kM$ exist. Give necessary and sufficient conditions on A and B for M to be zero.

Solution This problem was solved by Alex Heinis and Wim Hesselink. As Wim Hesselink sent in a solution first, he will receive the prize. The following is based on both solutions. Clearly the matrix M is unique, if it exists. For any linear map $f\colon \mathbf{C}^n \to \mathbf{C}^n$, the vector space $\mathbf{C}^n$ is the direct sum of the generalized eigenspaces $E_{f,\lambda} = \ker(f-\lambda)^n$ for eigenvalues λ of f by the theory of Jordan normal forms.

Lemma 1. For any linear map f on $\mathbf{C}^n$ we have $\lim_k f^k = 0$ if and only if every eigenvalue λ of f satisfies $|\lambda| < 1$.

Proof. The only-if part being obvious, we assume that every eigenvalue λ of f satisfies $|\lambda| < 1$. Let λ be such an eigenvalue. Then the restriction f_λ of f to the generalized eigenspace $E_{f,\lambda}$ can be written as $\lambda \cdot \mathrm{id} + m$, with $m^n = 0$. We get $f_\lambda^k = \sum_{j<n} \binom{k}{j}\lambda^{k-j}m^j$, which tends to 0, because in each term $\binom{k}{j}$ only grows polynomially in k. We conclude that f tends to 0. □

Lemma 2. Every eigenvalue λ of A or B satisfies $|\lambda| < 1$ or $\lambda = 1$.

Proof. For two sequences $(X_k)_k$ and $(Y_k)_k$ of matrices we write $X_k \sim Y_k$ if $\lim_k(X_k - Y_k) = 0$. Set $C = \lim_k(A^k + B^k)$. Then we have

$$C - A^{k+1} \sim B^{k+1} \sim B(C - A^k) = BC - BA^k,$$

and therefore $\lim_k(B-A)A^k = (B-I)C$. Let $x \in \mathbf{C}^n$ be an eigenvector for A with eigenvalue λ. Then $\lambda^k(Bx - \lambda x) = (B-A)A^k x$ converges, namely to $(B-I)Cx$. We conclude that either $|\lambda| < 1$ or $\lambda = 1$, or $Bx = \lambda x = Ax$, in which case $2\lambda^k x = (A^k + B^k)x$ converges, and we also find $|\lambda| < 1$ or $\lambda = 1$. The statement for eigenvalues of B follows from symmetry. □

Let a and b denote the linear maps on $\mathbf{C}^n$ defined by multiplication by A and B respectively, whose eigenvalues are given in the previous lemma. Let $p, m, r\colon \mathbf{C}^n \to \mathbf{C}^n$ be the unique linear maps that equal 0, 0, and a, respectively, on the generalized eigenspaces $E_{a,\lambda}$ of a associated to eigenvalues λ with $|\lambda| < 1$, while their restrictions to $E_{a,1}$ equal id, $a - \mathrm{id}$, and 0 respectively. In other words, with respect to the decomposition

$$\mathbf{C}^n \cong E_{a,1} \oplus \left(\bigoplus_{|\lambda|<1} E_{a,\lambda} \right)$$

the maps p, m, r are given as

$$p = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad m = \begin{pmatrix} a-\mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Then m is nilpotent, we have $a = p + m + r$, and the identities

$$p^2 = p, \quad pm = m = mp, \quad pr = mr = 0 = rm = rp, \quad \lim_k r^k = 0$$

hold, the latter by Lemma 1. Similarly, we may write $b = q + l + s$ where l is nilpotent and

$$q^2 = q, \quad ql = l = lq, \quad qs = ls = 0 = sl = sq, \quad \lim_k s^k = 0.$$

Of course the decomposition in generalised eigenspaces for a and b are not necessarily the same. We have

$$a^k + b^k = r^k + s^k + p + q + \sum_{1\le j<n} \binom{k}{j}(l^j + m^j).$$

Redactie:
Johan Bosman
Gabriele Dalla Torre
Ronald van Luijk
Lenny Taelman

Problemenrubriek NAW
Mathematisch Instituut
Postbus 9512, 2300 RA Leiden
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

As this has a limit and $\lim_k r^k = \lim_k s^k = 0$, we find that all terms $l^j + m^j$ for $1 \leq j \leq n$ vanish. In particular, $l + m = 0$ and $l^2 + m^2 = 0$, so that $l = -m$ and $m^2 = 0$. We conclude $a^k = p + r^k + km$ and $b^k = q + s^k - km$, so that $\lim_k a^k - km = p$ and $\lim_k b^k + km = q$. The first statement of the problem follows for the matrix M associated to m. The matrix M is zero if and only if the restriction of a to the generalized eigenspace $E_{a,1}$ is the identity.

Problem (Star) 2008-2/11 Let V be the complex vector space of all functions $f : \mathbf{C} \to \mathbf{C}$. Let W be the smallest linear subspace of V with the properties:

- the function $f(z) = z$ belongs to W,
- for all $f \in W$, $|f| \in W$.

Does $f(z) = \overline{z}$ belong to W?

Solution The following solution is due to David Preiss (Warwick), and was communicated to us by Miklos Laczkovich. Since the solution was already known, there is no prize winner.

We will show that $f(z) = \overline{z}$ does not belong to W. We claim that it suffices to show that there is a complex vector space S of complex valued functions on the circle $\mathbf{R}/2\pi\mathbf{Z}$ with the properties that

- $h(x) = e^{ix}$ belongs to S,
- for all $h \in S$, $|h| \in S$,
- $h(x) = \cos(x)$ does not belong to S.

Indeed, if the function $f(z) = \overline{z}$ is in W, then the function

$$\frac{e^{ix} + f(e^{ix})}{2} = \cos(x)$$

belongs to S.

Construction of S. Let U be the family of regions $U = \{x + iy : \psi(x) < y < \phi(x)\}$, where $\phi, \psi : \mathbf{R} \to \mathbf{R}$ are continuous, $\psi \leq 0 < \phi$ and $\{x : \psi(x) = 0\}$ is locally finite in $\mathbf{R}$.

Let H be the set of functions F on $\mathbf{C}$ for which there is a region $U \in$ U so that F is holomorphic on U, and such that there is an $a < 1$ with $\limsup_{z \in U, |z| \to \infty} |F(z)|/|z|^a = 0$.

Let F be the set of continuous functions $f : \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}$ with the property that there exist a positive integer n and functions $F_1, \ldots, F_n \in$ H (with corresponding regions $U_1, \ldots, U_n \in$ U) and open intervals $I_1, \ldots, I_n$ covering the circle minus a finite number of points so that $\cos x \in U_j$ and $f(x) = F_j(\cos x)$ for all $1 \leq j \leq n$ and all $x \in I_j$.

Let S be the set of all functions $\mathbf{R}/2\pi\mathbf{Z} \to \mathbf{C}$ of the form $f + ig + ce^{ix}$ where $f, g \in$ F and $c \in \mathbf{C}$. The set S is a linear subspace of the complex vector space of all complex-valued functions on the circle.

Clearly $h(x) = e^{ix}$ is an element of S.

Proof that S *is closed under* $h \mapsto |h|$. Let h be a function in S, and write h as

$$h(x) = F_j(\cos x) + iG_j(\cos x) + ce^{ix}$$

on the open interval I_j, with $F_j, G_j \in$ H. Let $a_j < 1$ be such that

$$\limsup_{z \in U_j, |z| \to \infty} |F(z)|/|z|^{a_j} = 0 \text{ and } \limsup_{z \in U_j, |z| \to \infty} |G(z)|/|z|^{a_j} = 0.$$

Assuming, as we may, that $\sin x$ does not change sign on any I_j, we have that on each I_j, $|h(x)|^2 = H_j(\cos x)$ where H_j is a linear combination of 1, F_j^2, G_j^2, $F_j(z)z$, $F_j(z)\gamma(z)$, $G_j(z)z$, $G_j(z)\gamma(z)$, where γ is a suitable branch of $\sqrt{1 - z^2}$. Removing from I_j the finite set where $h(x) = 0$ we have that on each remaining interval $|h|$ coincides with a branch of $H_j^{1/2}$ and one verifies that $|h| \in$ F, where the constant can be taken to be $(a + 1)/2$.

Proof that $h(x) = \cos x$ *does not belong to* S. Assume that $\cos x = f(x) + ig(x) + ce^{ix}$ where $c \in \mathbf{C}$ and $f, g \in$ F. Writing $c = u + iv$ and using that f, g are real, we get $f(x) = (1 - u)\cos x + v \sin x$, $g(x) = -v \cos x - u \sin x$. For any interval I on which we can use the definition of $f, g \in$ F (and on which $\sin x \neq 0$) we therefore have $F, G \in$ H and $U \in$ U so that $F(z) = (1 - u)z + v\gamma(z)$ and $G(z) = -vz - u\gamma(z)$, where γ is a branch of $\sqrt{1 - z^2}$ on U. We have that

$$\limsup_{|z| \to \infty, z \in U} \frac{|F(z)|}{|z|} = \limsup_{|z| \to \infty, z \in U} \frac{|((1 - u)z + v\gamma(z))|}{|z|} = \sqrt{(1 - u)^2 + v^2}.$$

Since this limit has to be zero we conclude that $u = 1$ and $v = 0$. A similar argument for G gives that $u = v = 0$, a contradiction.

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| Problem Section

Problem (Star) 2008-2/1 Let the continuous function $f_1 : (0,1] \to \mathbf{C}$ be such that

$$\int_0^1 f_1(t)dt$$

exists (and is finite) as an improper Riemann integral. Prove that f_1 has a unique extension to $f : \mathbf{R}^+ \to \mathbf{C}$ that is

- continuous on $\mathbf{R}^+$,
- differentiable on $(1,\infty)$ and satisfies the differential-difference equation

$$f'(x) = -\frac{1}{x} f(x-1) \qquad (x > 1). \tag{1}$$

Also, determine

$$\lim_{x\to\infty} x f(x).$$

Finally, show that, if $\int_0^1 f_1(t)dt = f_1(1)$, then the series $\sum_{n=0}^{\infty} n f(n)$ and the integral

$$\int_0^\infty f(t)dt$$

both converge absolutely and have the same value.

Solution We received solutions from Joris Bierkens and J. Arias de Reyna & J. van de Lune. Joris Bierkens will receive the prize.
The following solution is based on the one given by Bierkens.
Define the functions $f_n : (n-1,n] \to \mathbf{C}$ inductively, by

$$f_n(x) := f_{n-1}(n-1) - \int_{n-1}^{x} \frac{1}{t} f_{n-1}(t-1)dt,$$

and glue them to a function f on $\mathbf{R}^+$. By the properties of the Riemann integral, this f is continuous on $\mathbf{R}^+$ and differentiable on $(1,\infty)$ and it satisfies the differential-difference equation (1). If $g : \mathbf{R}^+ \to \mathbf{C}$ is another function with these properties, then we see that $g'(t) = f'(t)$ on $(1,2]$. From $f(1) = g(1)$ we conclude $f = g$ on $(1,2]$. Repeating this argument it follows that $g = f$ everywhere on $\mathbf{R}^+$.
In order to determine $\lim_{x\to\infty} x f(x)$, note that (1) implies

$$(xf(x))' = f(x) - f(x-1). \tag{2}$$

Therefore

$$\lim_{x\to\infty} x f(x) = f(1) + \int_1^\infty (xf(x))' dx = f_1(1) - \int_0^1 f_1(x)dx,$$

provided that the limit $\lim_{x\to\infty} x f(x)$ exists.
For the last part of the problem, integrate (2) to obtain the recursion

$$\int_n^{n+1} f(t)dt = \int_{n-1}^{n} f(t)dt + (n+1)f(n+1) - nf(n) \qquad (n \geq 1).$$

Now suppose $\int_0^1 f_1(t)dt = f_1(1)$. This recursion implies

Redactie:
Johan Bosman
Gabriele Dalla Torre
Ronald van Luijk
Lenny Taelman

Problemenrubriek NAW
Mathematisch Instituut
Postbus 9512, 2300 RA Leiden
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

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$$\int_{n-1}^{n} f(t)dt = nf(n). \tag{3}$$

We have for $n > 1$

$$\begin{aligned}\int_{n}^{n+1} |f(t)|dx &= \int_{n}^{n+1} \left| f(n) - \int_{n}^{x} \frac{1}{t} f(t-1)dt \right| dx \\ &\le |f(n)| + \int_{n}^{n+1} \frac{1}{n} \int_{n}^{n+1} |f(t-1)|dtdx \\ &\le \frac{1}{n} \int_{n-1}^{n} |f(t)|dt + \frac{1}{n} \int_{n}^{n+1} |f(t-1)|dt = \frac{2}{n} \int_{n-1}^{n} |f(t)|dt.\end{aligned}$$

So, by the ratio test, the series

$$\sum_{n=1}^{\infty} \int_{n-1}^{n} |f(t)|dt$$

converges. Since we have

$$\sum_{n=1}^{\infty} |nf(n)| = \sum_{n=1}^{\infty} \left| \int_{n-1}^{n} f(t)dt \right| \le \int_{0}^{\infty} |f(t)|dt = \sum_{n=1}^{\infty} \int_{n-1}^{n} |f(t)|dt,$$

we conclude that both $\sum_{n=0}^{\infty} nf(n)$ and $\int_0^\infty f(t)dt$ converge absolutely, and from (3) it follows that they have the same limit.

Problem (Star) 2008-2/4 Let $p\colon [0,1] \to \mathbf{R}$ be a continuous function with $p(t) \ge 0$ for all $t \in [0,1]$ and $\int_0^1 p(t)dt = 1$. Does the function $f\colon \mathbf{C} \to \mathbf{C}$ given by

$$f(z) = e^z - \int_0^1 p(t)e^{zt}dt$$

have infinitely many zeroes?

Solution We received solutions from R.A. Kortram and J. Arias de Reyna & J. van de Lune. R.A. Kortram will receive the prize.
The following solution is based on the one given by Kortram.
We shall prove that the answer is 'yes'. The function f has a Taylor series expansion given by

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{z^n}{n!}$$

with $a_n = 1 - \int_0^1 t^n p(t)dt$. The coefficients a_n are real and satisfy $0 < a_n < 1$ so for all $r \in \mathbf{R}_{>0}$ we have

$$M_r(f) := \max_{|z|=r} |f(z)| = f(r) < e^r. \tag{4}$$

This shows that f is of order (at most) 1: the order of the entire function f is the infimum of all m such that $f(z) = O(e^{|z|^m})$ as $z \to \infty$.
From now on, assume that f has only finitely many zeroes $z_1, \ldots, z_N$ with multiplicities $e_1, \ldots, e_N$. Hadamard's factorization theorem tells us how an entire function of given order can be expressed as product in terms of its zeroes and leads in our case to

$$f(z) = \phi(z) e^{\lambda z + \mu} \quad \text{with} \quad \phi(z) = \prod_{j=1}^{N} (z - z_j)^{e_j}$$

for certain $\lambda, \mu \in \mathbf{C}$. Since the Taylor coefficients of f are real, we have $\phi(z) \in \mathbf{R}[z]$, $\lambda \in \mathbf{R}$ and $e^{\mu} \in \mathbf{R}$ and hence there is a real number c with

$$f(z) = c\phi(z) e^{\lambda z}.$$

Now put $g(z) = \int_0^1 p(t) e^{zt} dt = e^z - f(z)$. We have

$$g(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} \quad \text{with} \quad b_n = \int_0^1 t^n p(t) dt > 0.$$

Hence for $r \in \mathbf{R}_{>0}$ we have

$$M_r(g) := \max_{|z|=r} |g(z)| = g(r) = e^r - f(r) = e^r - c\phi(r) e^{\lambda r}.$$

The fact that $f(0) = 0$ implies $\deg(\phi) \geq 1$; combining this with (4) we get $\lambda < 1$. So there is an $R \in \mathbf{R}$ such that for all $r > R$ we have $M_r(g) > e^r/2$.
Choose $\varepsilon < 1/4$ and $\delta \in [0, 1)$ with $\int_\delta^1 p(t) dt < \varepsilon$. Then also $\int_\delta^1 t^n p(t) dt < \varepsilon$. Choose K with $\delta^K < \varepsilon$. For $n \geq K$ we have

$$\int_0^\delta t^n p(t) dt \leq \delta^n \int_0^\delta p(t) dt \leq \delta^n \int_0^1 p(t) dt < \varepsilon$$

and thus $b_n < 2\varepsilon$.
For $r \geq R$ we get the following inequality:

$$e^r/2 < M_r(g) = g(r) < \sum_{n=0}^{K-1} b_n \frac{r^n}{n!} + 2\varepsilon \sum_{n=K}^{\infty} \frac{r^n}{n!} < \sum_{n=0}^{K-1} b_n \frac{r^n}{n!} + 2\varepsilon \cdot e^r,$$

which is a contradiction for large r.

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Problem (Star) 2008-2/7 For $n = 1, 2, 3, \ldots$ we define the function $\Phi_n : \mathbf{R} \to \mathbf{R}$ by

$$\Phi_n(x) = (2n)^x - (2n-1)^x + (2n-2)^x - (2n-3)^x + \cdots + 2^x - 1.$$

Prove or disprove that for all $x \in \mathbf{R}$ and for all n
1. $\Phi'_n(x) > 0$;
2. $\Phi''_n(x) > 0$.
What can be said about higher derivatives?

Solution We received an ingenious solution from Juan Arias de Reyna and Jan van de Lune, who are awarded the prize. They show that Φ'_n and Φ''_n are strictly positive, but that there is an x so that $\Phi'''_n(x) < 0$. We limit ourselves to giving a sketch of their solution.
Proof that Φ'_n and Φ''_n are strictly positive.
First of all, we may restrict ourselves to $x < 0$, since for $x \geq 0$ it is clear that all the derivatives of $\Phi_n(x)$ are positive.
Denote the first derivative of $-\Phi_n(-x)$ by ϕ_n and the second derivative of $\Phi_n(-x)$ by ψ_n. We need to show that

$$\phi_n(x) = \frac{\log 2}{2^x} - \frac{\log 3}{3^x} + \cdots - \frac{\log(2n-1)}{(2n-1)^x} + \frac{\log(2n)}{(2n)^x} > 0$$

and

$$\psi_n(x) = \frac{(\log 2)^2}{2^x} - \frac{(\log 3)^2}{3^x} + \cdots - \frac{(\log(2n-1))^2}{(2n-1)^x} + \frac{(\log(2n))^2}{(2n)^x} > 0$$

for all n and for all $x > 0$.
The theory of Dirichlet series gives an entire function $\eta(s)$ so that

$$\lim_{n\to\infty} \phi_n(s) = \eta'(s)$$

and

$$\lim_{n\to\infty} \psi_n(s) = -\eta''(s)$$

for all positive real s. (For $s > 1$ this is trivial, by the absolute convergence of the series $\sum_n (-1)^n n^{-s}$.) In fact $\eta(s) = (1 - 2^{1-s})\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function.
Assume that there exists an n and an $x > 0$ so that $\phi_n(x)$ (resp. $\psi_n(x)$) is non-positive. It is not too hard to show that this implies that

$$\eta'(x) \leq 0$$

respectively

$$\eta''(x) \geq 0.$$

It therefore suffices to show that $\eta'(x) > 0$ and $\eta''(x) < 0$ for all $x > 0$.
First, one shows that for all x such that

$$x > 2\frac{\log(\log(3)) - \log(\log(2))}{\log(3) - \log(2)} \approx 2.2718$$

and all $k > 0$, one has

$$\frac{(\log(2k))^2}{(2k)^x} \geq \frac{(\log(2k+1))^2}{(2k+1)^x}.$$

If follows that $\eta''(x)$ is negative for all $x > 2.2718$.
The rest of the argument depends on the following inequality

Redactie:
Johan Bosman
Gabriele Dalla Torre
Ronald van Luijk
Lenny Taelman
Problemenrubriek NAW
Mathematisch Instituut
Postbus 9512, 2300 RA Leiden
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

$$|\eta'''(x)| \leq B(s) := \frac{3+6s}{(1+s)^3} \qquad (x \geq s > 0), \tag{1}$$

the proof of which we postpone.
Now one verifies numerically that

$$\eta''(0) \approx -0.06103 < 0,$$

so that by the maximal slope principle and the inequality (1) one finds

$$\eta''(x) < 0, \text{ for all } x < \frac{-\eta''(0)}{B(0)} \approx 0.020343$$

Repeating this about 20 times one finds $\eta''(x) < 0$ for all x between 0 and 2.28, from which we conclude that $\eta''(x) < 0$ for all positive x, this finishes the proof for the second derivative.
For the first derivative, observe that since $\eta'' < 0$ we have that η' is strictly decreasing. But it is easy to check that $\eta'(x)$ is positive for all x sufficiently large, therefore $\eta'(x) > 0$ for all x.
Proof of (1). We now sketch how to prove the crucial inequality (1).
Let $E\colon \mathbf{R} \to \mathbf{R}$ be the "triangle wave" function of period 2 which satisfies $E(x) = \frac{2x-1}{4}$ for $0 \leq x \leq 1$ and $E(x) = \frac{3-2x}{4}$ for $1 \leq x \leq 2$. One shows that

$$\eta(s) = \frac{1}{2} + \frac{s}{4} + s(s+1)\int_1^\infty \frac{E(x)}{x^{s+2}}dx.$$

Computing the third derivative of this, and using $|E(x)| \leq \frac{1}{4}$ one finds

$$|\eta'''(s)| \leq \frac{3+6s}{(1+s)^3}$$

and the desired inequality follows by noting that the right-hand side is decreasing for $s > 0$.
Higher derivatives. If Φ_n''' were strictly positive for all $x > 0$ then it would follow that $\eta'''(x) \geq 0$ for all $x \geq 0$. But one can verify numerically that

$$\eta'''(0) \approx -0.02347468,$$

a contradiction. ∎