Problemen

| Problem Section

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Edition 2008/1
We received submissions from Johan de Ruiter (Leiden), H. Reuvers (Maastricht), R.A. Kortram (Nijmegen), Jaap Spies, Paolo Perfetti (Dipartimento di Matematica, Roma 2, Rome), Hans Montanus, Noud Aldenhoven & Daan Wanrooy (Nijmegen), Sander Kupers (Utrecht), Jacky Chow (Sydney) en Thijmen Krebs (Delft).

Problem 2008/1-A Denote the fractional part of a positive real number x by $\{x\}$, for example $\{\pi\} = \pi - 3$. Evaluate the following double integral:

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} dxdy.$$

Solution We received correct solutions from Johan de Ruiter, H. Reuvers, R.A. Kortram, Jaap Spies, Paolo Perfetti, Hans Montanus, Noud Aldenhoven & Daan Wanrooy, Sander Kupers, Jacky Chow en Thijmen Krebs.
All submitted solutions where similar to one another. The book voucher goes to Noud Aldenhoven and Daan Wanrooy, whose names were pulled out of a hat due to the difficulty of splitting one 20 euro voucher into 10 equal redeemable parts (the lowest existing denomination being 5 euro).
Let Δ_n be the triangle bounded by $x = 1$, $y = x/n$ and $y = x/(n+1)$. Let A_n be the area of Δ_n. Then we have

$$\int_{\Delta_n} \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} dxdy = A_n - \frac{n}{4}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right).$$

The same holds for the triangle obtained by interchanging the roles of x and y. Summing over all triangles we obtain

$$\begin{aligned}\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} dxdy &= 1 - 2\sum_{n=1}^{\infty} \frac{n}{4}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= 1 - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2} + \frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{\pi^2}{12}\end{aligned}$$

Problem 2008/1-B Let S be a set consisting of 15 integers, and such that for all $s \in S$ there exist $a, b \in S$ with $s = a + b$.

1. Show that there exists a non-empty subset $T \subset S$ of at most seven elements that add up to 0.
2. Show that this does not need to be true for S with 16 elements.

Solution We received no submissions for problem B. This is not so surprising as our own solution turned out to be incorrect. However, we also do not have a counterexample. Therefore, this problem has been promoted to the starred problem above.

Problem 2008/1-C Let $f : \mathbf{R} \to \mathbf{R}$ be a C^∞ function (that is, all higher derivatives of f exist and are continuous) such that

1. $f(x) = 0$ if $x \le 0$,
2. $f(x) > 0$ if $x > 0$.

Oplossingen | Solutions

Is it true that $\sqrt{f} : \mathbf{R} \to \mathbf{R}_{\geq 0}$ is a C^1 function (that is, that its derivative exists and is continuous)?

Solution We received a correct solution from R.A. Kortram. Several people made the mistake of using l'Hôpital's rule where it is not applicable.
Bastien Marmeth from Rennes pointed out to us that this problem is solved in the literature: *see e.g.* J. Dieudonné, *Sur un théorème de Glaeser*, J. Anal. Math 23, 1970.

The following solution is based on R.A. Kortram's.
Put $g(x) := \sqrt{f(x)}$. It is not too hard to show that g is differentiable everywhere, continuously differentiable outside $x = 0$ and that $g'(x) = 0$ for all $x \leq 0$. It remains to show that

$$\lim_{x \to 0, x > 0} g'(x) = 0,$$

or, equivalently, that

$$\lim_{x \to 0, x > 0} \frac{f'(x)^2}{f(x)} = 0. \tag{1}$$

This is the crux of the problem.
If there is a positive δ such that $f'(x) \neq 0$ for all $x \in (0, \delta)$ then by *l'Hôpital* we have that

$$\lim_{x \to 0, x > 0} \frac{f'(x)^2}{f(x)} = \lim_{x \to 0, x > 0} \frac{2f'(x)f''(x)}{f'(x)} = 2f''(0) = 0.$$

If there is no such δ we proceed as follows. Let $\epsilon > 0$ be given and choose a $d > 0$ such that $|2f''(x)| < \epsilon$ on the interval $(0, d)$ and such that $f'(d) = 0$. We claim that

$$\frac{f'(x)^2}{f(x)} < \epsilon$$

for all $x \in (0, d)$, which suffices to show (1).
Indeed, consider the open set

$$U := \{x \in (0, d) : f'(x) \neq 0\}.$$

It is a countable union of disjoint intervals (p_n, q_n). Note that $f'(p_n) = f'(q_n) = 0$ and that f' has a constant sign on (p_n, q_n), so f is either monotonously increasing or decreasing on (p_n, q_n). Assume that f is increasing on (p_n, q_n) (the other case is similar). Then for all $x \in (p_n, q_n)$ we have that

$$\frac{f'(x)^2}{f(x)} < \frac{f'(x)^2}{f(x) - f(p_n)} = \frac{f'(x)^2 - f'(p_n)^2}{f(x) - f(p_n)}.$$

But by *Cauchy's mean value theorem* applied to the functions $f'(x)^2$ and $f(x)$ there is a point $t \in (p_n, x)$ such that

$$\frac{f'(x)^2 - f'(p_n)^2}{f(x) - f(p_n)} = \frac{2f''(t)f'(t)}{f'(t)}$$

and the latter is bounded by ϵ by the definition of d.